K3 surfaces and moduli of holomorphic differentials

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Abstract

In this thesis we investigate the birational geometry of various moduli spaces; moduli spaces of curves together with a k-differential of prescribed vanishing, best known as *strata of differentials*, moduli spaces of K3 surfaces with marked points, and moduli spaces of curves. For particular genera, we give estimates for the Kodaira dimension, construct unirational parameterizations, rational covering curves, and different birational models.

In Chapter 1 we introduce the objects of study and give a broad brush stroke about their most important known features and open problems. In Chapter 2 we construct an auxiliary moduli space that serves as a bridge between certain finite quotients of $\mathcal{M}_{g,n}$ for small g and the moduli space of polarized K3 surfaces of genus eleven. We develop the deformation theory necessary to study properties of the mentioned moduli space.

In Chapter 3 we use this machinery to construct birational models for the moduli spaces of polarized K3 surfaces of genus eleven with marked points and we use this to conclude results about the Kodaira dimension. We prove that the moduli space of polarized K3 surfaces of genus eleven with n marked points $\mathcal{F}_{11,n}$ is unirational when $n \leqslant 6$ and uniruled when $n \leqslant 7$. We also prove that $\mathcal{F}_{11,n}$ has non-negative Kodaira dimension for $n \geqslant 9$. In the final section, we make a connection with some of the missing cases in the Kodaira classification of $\overline{\mathcal{M}}_{g,n}$.

Finally, in Chapter 4 we address the question concerning the birational geometry of strata of holomorphic and quadratic differentials. We show strata of holomorphic and quadratic differentials to be uniruled in small genus by constructing rational curves via pencils on K3 and del Pezzo surfaces respectively. Restricting to genus $3 \leqslant g \leqslant 6$ we construct projective bundles over rational varieties that dominate the holomorphic strata with length at most $g - 1$, hence showing in addition, these strata are unirational.

Zusammenfassung

In dieser Arbeit behandeln wir die birationale Geometrie verschiedener Modulräume; die Modulräume von Kurven mit einem k-Differential mit vorgeschierbenen Nullen, besser bekannt als *Strata von Differenzialen*, Moduln von K3 Flächen mit markierten Punkten und Moduln von Kurven. Für bestimmte Geschlechter nennen wir Abschätzungen der Kodaira-Dimension, konstruieren unirationale Parametrisierungen, rationale deckende Kurven und unterschiedliche birationale Modelle.

In Kapitel 1 führen wir die zu untersuchenden Objekte ein und geben einen kurzen Überblick ihrer wichtigsten Eigenschaften und offenen Problemen. In Kapitel 2 konstruieren wir einen Hilfsmodulraum, der als Brücke zwischen bestimmten finiten Quotienten von $\mathcal{M}_{g,n}$ für kleines g und den Moduln der polarisierten K3 Flächen vom Geschlecht 11 dient. Wir entwickeln die Deformationstheorie, die nötig ist, um die Eigenschaften und die oben genannten Modulräume zu erforschen.

In Kapitel 3 bedienen wir uns dieser Werkzeuge, um birationale Modelle für Moduln polarisierter K3 Flächen vom Geschlecht 11 mit markierten Punkten zu konstruieren. Diese nutzen wir, um Resultate über die Kodaira-Dimension herzuleiten. Wir beweisen, dass der Modulraum von polarisierten K3 Flächen vom Geschlecht 11 mit n markierten Punkten $\mathcal{F}_{11,n}$ unirational ist, falls $n \leqslant 6$, und *uniruled*, falls $n \leqslant 7$. Wir beweisen auch, dass die Kodaira-Dimension von $\mathcal{F}_{11,n}$ nicht-negativ ist für $n \geqslant 9$. Im letzten Kapitel gehen wir noch auf die fehlenden Fälle der Kodaira-Klassifizierung von $\overline{\mathcal{M}}_{g,n}$ ein.

Schließlich behandeln wir in Kapitel 4 die birationale Geometrie mit Blick auf die Strata von holomorphen und quadratischen Differentialen. Wir zeigen, dass die Strata holomorpher und quadratischer Differentiale von niedrigem Geschlecht *uniruled* sind, indem wir rationale Kurven mit *pencils* auf K3 und del Pezzo Flächen konstruieren. Durch das Beschränken des Geschlechts $3 \leqslant g \leqslant 6$ bilden wir projektive Bündel über rationale Varietäten, die die holomorphe Strata mit maximaler Länge $g - 1$ dominieren. Also zeigen wir auch, dass diese Strata unirational sind.

Once when I lectured at the University of California at Fullerton, a student asked me for a short, simple definition of reality. I thought it over and answered, 'Reality is that which when you stop believing in it, it doesn't go away.'[1]

Déjense de preguntas.
En el lecho de muerte
cada uno se rasca con sus uñas.[2]

[1]Fragment in *VALIS*, Philip K. Dick. Ed. Gollancz, 2001. Collection SF Masterworks.

[2]Quit with the questions. / On the deathbed / everyone scratches with his own nails. Fragment in *Advertencia, De Versos de Salón*, Nicanor Parra. Chistes para desorientar a la poesía, ed. N. Alonso and G. Triviños, Colección Visor de Poesía.

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CHAPTER 1

Introduction

1.1 The moduli spaces $\mathcal{M}_{g,n}$, $\mathcal{H}_g^k(\mu)$ and $\mathcal{F}_{g,n}$

The main two objects that concern this thesis are (nodal) projective curves and projective K3 surfaces. Before going to moduli, let us set up the basic definitions. Our base field will be the field of complex numbers and a *variety* X over $\mathbb{C}$ will be an integral, separated $\mathbb{C}$-scheme of finite type. Following standard definitions, the variety X is said to be *complete* if the structure morphism

$$X \to \mathrm{Spec}\,(\mathbb{C})$$

is proper and, *projective*, if it is a closed subvariety of a projective space $\mathbb{P}^n_{\mathbb{C}}$.

Definition 1.1. A projective curve with at worst nodal singularities is called *stable* if the group of automorphisms is finite.

Definition 1.2. A K3 surface S is a compact connected complex manifold of dimension two, for which $\Omega_S^2 \cong \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. A *polarized* K3 surface *of genus* g is a pair (S, H), where S is an algebraic K3 surface and $H \in \mathrm{Pic}(S)$ is a primitive nef line bundle with

$$H^2 = 2g - 2.$$

1.1.1 Functors and moduli

This is standard set up in moduli theory, if the reader is familiar with the *functor of points* and *coarse representability*, we recommend to skip this short section.

Recall that, for a scheme X, its *functor of points* is defined to be the contravariant functor

$$X(\cdot) : \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$$

from the category of $\mathbb{C}$-schemes to the category of sets, defined at the level of objects as

$$X(B) = \mathrm{Hom}_{\mathbb{C}}(B, X)$$

and defined by post-composition at the level of morphisms;

$$\begin{array}{rcl} X(f : B \to B') : \mathrm{Hom}(B', X) & \to & \mathrm{Hom}(B, X) \\ \phi & \mapsto & \phi \circ f. \end{array}$$

A contravariant functor $F : \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ is said to be *representable* if there exist a scheme X, whose functor of points is isomorphic to F. One can check from the definition that, if such X exists, it is unique up to isomorphisms.

We to set up the most basic definition in moduli theory, which is the one of a *family*.

Definition 1.3. A *family* over a $\mathbb{C}$-scheme B is a flat projective morphism

$$\pi : \mathscr{X} \to B.$$

A *family with* n *markings* consists of a family π, together with n sections, that is, maps

$$s_1, \dots, s_n : B \to \mathscr{X}$$

such that the composition $\pi \circ s_i$ is the identity on B. Two families are isomorphic if they are B-isomorphic and two marked families are isomorphic if the B-isomorphism preserves the sections.

A moduli functor usually refers to a functor $M : \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ that assign to a scheme B, the set of isomorphism classes of families over B, with extra conditions. If a moduli functor M is representable by a scheme $\mathcal{M}$, that is, if there is an isomorphism of functors

$$\Phi : M \xrightarrow{\cong} \mathrm{Hom}(\cdot, \mathcal{M}),$$

then, every family $\mathscr{X} \to B$, up to isomorphism, corresponds uniquely to a map $B \to \mathcal{M}$ and every such map, $B \to \mathcal{M}$, gives rise to a family over B. In particular the identity

$$1 : \mathcal{M} \to \mathcal{M}$$

corresponds to a family

$$\mathscr{C} \to \mathcal{M}$$

that we call the *universal family* of the moduli space $\mathcal{M}$. Unfortunately, some of the most basic moduli functors are not representable. There are two ways of confronting this difficulty. One is to enlarge the category of schemes so the functor becomes representable; this leads to the theory of algebraic stacks. The other option is to loosen the conditions of representability hoping to find a scheme that captures most of the features of the moduli problem that we are treating. This leads to the notion of *coarse moduli*, the one that we will follow.

Definition 1.4. A moduli functor $M : \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ is said to be *coarsely represented* by a scheme $\mathcal{M}$, if there is a morphism of functors $\Phi : M \to \mathcal{M}(\cdot)$ such that

- $\Phi(\mathrm{Spec}\,(\mathbb{C})) : M(\mathrm{Spec}\,(\mathbb{C})) \to \mathcal{M}(\mathrm{Spec}\,(\mathbb{C}))$ is a bijection and
- for any other scheme $\mathcal{M}'$ and functor $\Phi' : M \to \mathcal{M}'(\cdot)$, there is a unique morphism $\Psi : \mathcal{M} \to \mathcal{M}'$ such that $\Phi' = \Psi \circ \Phi$.

Notice that the definition implies:

(1) If $\mathcal{M}$ exists, it is unique up to isomorphism.

(2) For every family $\mathscr{X} \to B$, one still has an induced map $B \to \mathcal{M}$.

(3) Closed points correspond to isomorphism classes of objects that we want to parametrize, e.g., curves of fixed genus, surfaces with fixed invariants, vector bundles with fixed slope on a fixed curve, polarized K3 surfaces of fixed genus or similar moduli problems.

Let us come back to our objects of interest.

Definition 1.5. Let $\pi : \mathscr{X} \to B$ be a family.

(1) A *family of stable (smooth) curves of genus* g is a family π, such that every fiber $X_b = \pi^{-1}(b)$ is a stable (smooth) curve of genus g.

(2) A *family of polarized K3 surfaces of genus* g is a family π, together with a line bundle $\mathscr{L}$ over $\mathscr{X}$, such that for every fiber, the pair $(X_b, \mathscr{L}|_{X_b})$ is a polarized K3 surface of genus g.

(3) Given positive integers k, g and an integer partition $\mu = (m_1, \ldots, m_n)$ of $k \cdot (2g-2)$, a *family of smooth curves of genus* g *together with canonical divisors of type* μ is a family of smooth curves with n markings $\pi : \mathscr{X} \to B$, $s_i : B \to \mathscr{X}$, such that, for some $L \in \mathrm{Pic}(B)$, the following relation holds in $\mathrm{Pic}(\mathscr{X})$:
$$\omega_\pi^{\otimes k} \cong \pi^* L \otimes \mathcal{O}_{\mathscr{X}} \left(\sum_{i=1}^{n} m_i s_{i*}[B] \right).$$

The moduli spaces $\mathcal{M}_{g,n}$, $\mathcal{F}_{g,n}$ and $\mathcal{H}_g^k(\mu)$ are the one that coarsely represent the following moduli functors:

- $M_{g,n}(B) := \{\text{families as in (1) with } n \text{ markings}\}/\text{iso}$,
- $F_{g,n}(B) := \{\text{families as in (2) with } n \text{ markings}\}/\text{iso}$,
- $H_g^k(\mu)(B) := \{\text{families as in (3)}\}/\text{iso}$.

When $n = 0$ we omit it in the notation.

Remark 1.6. The question of proving the existence of these moduli spaces is not the main concern of this thesis. Their existence, together with standard properties, such as separateness, quasi-projectivity, and normality, will be taken for granted. We refer to [ACG11] for the construction of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. From the existence of $\mathcal{M}_{g,n}$ and a *universal Picard variety*, follows the existence of $\mathcal{H}_g^k(\mu)$. The construction of $\mathcal{F}_g$ will be briefly treated in the next section §1.2. The existence of $\mathcal{F}_{g,n}$ follows from the existence of $\mathcal{F}_g$, see Remark 1.39.

1.2 Moduli of polarized K3 surfaces

The moduli space of polarized K3 surfaces has a surprising resemblance with the moduli space of abelian varieties. In both cases it can be constructed as a quotient of certain period domain by an arithmetic group. For K3 surfaces, using Hodge Theory, the construction of its moduli is considerably easier than the construction of $\mathcal{M}_g$.

A fundamental difference between K3 surfaces and curves is that an arbitrary K3 surface is not necessarily projective and we are forced to keep track of a polarization and its numerical invariants if we want to construct a moduli of algebraic K3 surfaces. Recall that for any compact Kähler manifold X, there is a decomposition (independent on the Kähler metric) of the complex vector space

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^q(X,\Omega_X^p).$$

We denote

$$H^{p,q}(X) = H^q(X,\Omega_X^p) \ \text{ and } \ h^{p,q} = \dim H^q(X,\Omega_X^p).$$

The integers $h^{p,q}$ are called *Hodge numbers*. We refer to [Voi02b] for an excellent introduction to Hodge Theory. Our starting point to construct the moduli space of K3 surfaces will be the Hodge diamond and the *K3 lattice*. For a full treatment we refer to [Huy16]. The Hodge diamond of a K3 surface S has the following shape

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \ = \ \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1. & & \end{matrix}$$

The cohomology group $H^2(S,\mathbb{Z})$, together with the cup-product $\langle \cdot, \cdot \rangle$, form a lattice isometric to a fixed lattice called the *K3 lattice*, denoted by Λ_{K3}. For those familiar with lattice theory, Λ_{K3} is non-degenerate, even, of rank 22 and signature $(3, 19)$. More explicitely,

$$\Lambda_{K3} = H^{\oplus 3} \oplus (-E_8)^{\oplus 2},$$

where H is the hyperbolic lattice $\mathbb{Z}^2$ (even, unimodular, of rank two and signature $(1, -1)$) and E_8 is the even, unimodular, positive definite lattice of rank 8 that corresponds to the Dynkin diagram E_8.

Definition 1.7. The two main sublattices of $H^2(S, \mathbb{Z})$ are the *Néron-Severi*

$$NS(S) = H^{1,1}(S) \cap H^2(S, \mathbb{Z})$$

and the *transcendental lattice*

$$T(S) \subset H^2(S, \mathbb{Z}),$$

defined to be the smallest sublattice such that $T(S) \otimes \mathbb{C}$ contains a generator of $H^{2,0}(S)$.

The first step towards the construction of a moduli of K3 surfaces is the following theorem.

Theorem 1.8 (Weak Torelli, cf. §VIII, Cor. 11.2 in [BHPV04]). *Two K3 surfaces S and S' are isomorphic if and only if there exist an isomorphism*

$$H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z}),$$

whose $\mathbb{C}$-extension $H^2(S, \mathbb{C}) \to H^2(S', \mathbb{C})$ is a Hodge isometry (that is, an isometry that preserves the Hodge decomposition).

Notice that $H^2(S, \mathbb{Z})$ is isometric to the K3 lattice Λ_{K3}.

Definition 1.9. A K3 surface with a *marking* is a pair (S, ϕ), where

$$\phi : H^2(S, \mathbb{Z}) \to \Lambda_{K3}$$

is an isometry.

Weak Torelli states that two K3 surfaces are isomorphic if and only if there are markings ϕ and θ such that $\theta^{-1}\phi$ is a Hodge isometry. Now we construct the global parameter space called *the period domain*. Consider the complex vector space $\Lambda_{K3} \otimes \mathbb{C}$ together with the $\mathbb{C}$-extension of the bilinear form $\langle \cdot, \cdot \rangle$. We define the *period domain of K3 surfaces* to be

$$\Omega_{K3} = \{[x] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid \langle x, x \rangle = 0 \text{ and } \langle x, \overline{x} \rangle > 0\}.$$

Notice that $\langle \lambda x, \overline{\lambda x} \rangle = \lambda\bar{\lambda}\langle x, \bar{x} \rangle$ and $\lambda\bar{\lambda} > 0$. The space is well defined and consist of an analytic open subset of the 20-dimensional smooth projective quadric defined by $\langle \cdot, \cdot \rangle$.

Let $\phi : H^2(S, \mathbb{C}) \to \Lambda_{K3} \otimes \mathbb{C}$ be the $\mathbb{C}$-extension of a marking on S. Recall that, the cup-product gets identified with the wedge product after the Hodge isomorphism and, for any non trivial $(2,0)$-form $\sigma \in H^{2,0}(S)$,

$$\sigma \wedge \sigma = 0 \quad \text{and} \quad \sigma \wedge \bar{\sigma} > 0.$$

In other words, the line $\phi\left(H^{2,0}(S)\right)$ in $\Lambda_{K3} \otimes \mathbb{C}$ correspond to a point on the period domain. We want the image of the transcendental lattice to determine the marking ϕ. The *period map* is the set theoretic map defined by

$$\begin{array}{rcl} \{\text{K3's with marking}\} & \to & \Omega_{K3} \\ (S, \phi) & \mapsto & \mathbb{P}\left(\phi(H^{2,0}(S))\right). \end{array}$$

As a consequence of Weak Torelli 1.8 and the fact that there are no non-trivial automorphisms on a K3 surface inducing the identity on $H^2(S, \mathbb{Z})$, cf. [BHPV04, §VIII, Prop. 11.3], one deduces injectivity of the period map. Surjectivity is more involved, but also holds.

Proposition 1.10 (§VIII, Cor. 14.2 in [BHPV04])**.** *The period map is bijective.*

Recall that, one of the quintessential properties of a coarse moduli space is that families of the desired object induce a map to the moduli space. If we want to understand the local structure of our moduli space or construct it by gluing local charts, we need to consider families $\pi : \mathscr{S} \to \triangle$ over a small disc. The choice of a marking $\phi : H^2(S, \mathbb{Z}) \to \Lambda_{K3}$ in families corresponds to the choice of a trivialization

$$\Phi : R^2\pi_*\mathbb{Z} \to \underline{\Lambda_{K3}}_{\triangle},$$

where the object on the right is the constant sheaf on $\triangle$ with value Λ_{K3}. This induces a map

$$\begin{array}{rcl} \mathscr{P}_\pi : \triangle & \to & \Omega_{K3} \\ t & \mapsto & \Phi_t\left(H^{2,0}(S_t)\right). \end{array}$$

Theorem 1.11 (Local Torelli, cf. §VIII, Thm. 7.3 in [BHPV04])**.** *For any marked K3 surface (S, ϕ), the map $\mathscr{P}_\pi$ is a local isomorphism when π is the versal deformation of (S, ϕ).*

Notice that, points on the period domain Ω_{K3} correspond to marked K3 surfaces, different markings on the same K3 surface correspond to automorphisms and families of marked K3 induce a map to the period domain. It would be enough to quotient Ω_{K3} by the group Γ of isometries of Λ_{K3} to have constructed a moduli space of K3 surfaces. The issue is that the action of Γ on the period domain is not properly discontinuous giving rise to a non-Hausdorff space. The existence of a universal family of marked K3 surfaces still holds and can be constructed using the local Torelli theorem, but the base is not well behaved. We would like to have at least a separated moduli. The solution to this problem is to add to the marking the data of a Kähler class $\kappa \in H^{1,1}(S, \mathbb{R})$ and a fixed image for it under the marking.

Definition 1.12. An *isomorphism of polarized K3 surfaces*, $f : (S, H) \to (S', H')$ is an isomorphism $f : S \to S'$ such that $f^*H' = H$. In particular, they have the same genus.

And the following theorem is the main building block in the construction of the moduli space of polarized K3 surfaces.

Theorem 1.13 (Strong Torelli, cf. §VIII, Thm. 11.1 and Cor. 11.4 in [BHPV04])**.** *For any two polarized K3 surfaces of the same genus* (S, H) *and* (S', H')*, if there is a Hodge isometry* $\phi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ *with* $\phi(H) = H'$*, then there exists a unique isomorphism* $f : S \to S'$ *with* $\phi = f^*$*.*

We fix $\theta \in \Lambda_{K3}$ with $\langle \theta, \theta \rangle = 2g - 2$. A *marked polarized K3 of genus* g is a marked K3 surface (S, ϕ) such that $(S, \phi^{-1}(\theta))$ is a polarized K3 of genus g. Notice that for any $\sigma \in H^{2,0}(S)$,

$$\langle \phi(\sigma), \theta \rangle = 0.$$

Thus, the period point of (S, ϕ) lies in the 19-dimensional refined period domain

$$\Omega_\theta := \{[x] \in \Omega_{K3} \mid \langle x, \theta \rangle = 0\}.$$

If $\Gamma(\theta)$ is the group of isometries of Λ_{K3} that fix θ, then it acts properly discontinuous on Ω_θ and by the Torelli theorems together with the surjectivity of the period map, one concludes:

Theorem 1.14 (§VIII, Thm 22.4 in [BHPV04])**.** *The quotient*

$$\mathcal{F}_g := \Gamma(\theta) \backslash \Omega_\theta$$

is independent of the choice of θ *and forms a coarse moduli space for polarized K3 surfaces of genus* g*.*

1.2.1 Comments on compactifications

For the moduli of polarized K3 surfaces $\mathcal{F}_g$, there are several well known ways of constructing compactifications. Probably, the most famous one is due to Baily and Borel, developed in [BB66], where they construct a general method to compactify arithmetic quotients of bounded symmetric domains. In this case, the boundary is the union of zero and one dimensional components with some modular meaning (corresponding to type II and type III degenerations), cf. [Sca87]. The downside is that the resulting space is heavily singular. The boundary is far too small to expect something different. Another well known way to compactify $\mathcal{F}_g$ is via *toroidal compactifications*, originally constructed by Ash-Mumford-Rapoport-Tai in [AMRT75] and Namikawa in [Nam80]. These are realized as toroidal blow-ups of the boundary of the Baily-Borel compactification. The disadvantage is that it is non-canonical, in the sense that many choices are involved in the construction and there is no standard modular meaning for points in the boundary. The advantage of toroidal compactifications is that, for certain choices, one can ensure that the resulting space has canonical singularities. This was used in [GHS07] to compute the Kodaira dimension of $\mathcal{F}_g$ for all, but finitely many g's.

Over $\mathcal{F}_g$ sits a natural $\mathbb{P}^g$-bundle $\mathcal{P}_g$, whose fiber over a point (S, H) in $\mathcal{F}_g$ is given by the linear system $|H|$. In principle, one could compactify $\mathcal{P}_g$ using the very general construction developed by Kollár, Shepherd-Barron and Alexeev in [KS88], [Ale96] and [Ale06]. This tool has its origin in the log-minimal model program and provides a method to compactify moduli spaces of surfaces of log-general type without making any choices and, in our case, the the boundary correspond to moduli of degenerate K3s together with a divisor on them. Anyhow, this compactification does not come with an explicit description of the boundary, which makes its study very hard. For small g, one should add to the list the *GIT* compactification. For instance, in [Sha80] is constructed the GIT compactification for $\mathcal{F}_2$ and studied its relation with the respective Baily-Borel compactification. Recently, Laza in [Laz16] compared the KSBA and GIT compactifications of $\mathcal{F}_2$ and Laza-O'Grady in [LO16]-[LO17], carried a full comparison between GIT, and Baily-Borel compactifications for $\mathcal{F}_3$. As the reader can notice, the differences and relations among different compactifications for $\mathcal{F}_g$ is subject of current research and we have partial answers just for a few cases.

1.3 Strata of k-differentials

Let $g \geqslant 2$ be an integer and $\mu = (m_1, \ldots, m_n)$ be a partition of $k \cdot (2g-2)$. As we saw in previous sections, the moduli space of k-canonical divisors of type μ is defined as the closed substack $\mathcal{H}_g^k(\mu) \subset \mathcal{M}_{g,n}$, whose set-theoretic description is given by

$$\mathcal{H}_g^k(\mu) := \left\{ [C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n} \mid \mathcal{O}_C\left(\sum_{i=1}^{n} m_i p_i\right) \cong \omega_C^{\otimes k} \right\}.$$

The scheme structure of this space can be constructed as follows. Let

$$\mathscr{J}ac_g^{k\cdot(2g-2)} \to \mathcal{M}_{g,n}$$

be the universal jacobian over $\mathcal{M}_{g,n}$ of degree $k \cdot (2g-2)$ and

$$\mathcal{M}_{g,n} \stackrel{\omega^{\otimes k}}{\to} \mathscr{J}ac_g^{k(2g-2)}$$

the k-th canonical section. The Abel-Jacobi map associated to μ is given by

$$\begin{array}{rcl} A_\mu : \mathcal{M}_{g,n} & \to & \mathscr{J}ac_g^{k(2g-2)} \\ [C, x_1, \dots, x_n] & \mapsto & [C, x_1, \dots, x_n, \mathcal{O}_C\left(\sum m_i x_i\right)] . \end{array}$$

The space $\mathcal{H}_g^k(\mu)$ is defined as the fiber product

$$\begin{tikzcd} \mathcal{H}_g^k(\mu) \arrow[r] \arrow[d] & \mathcal{M}_{g,n} \arrow[d, "A_\mu"] \\ \mathcal{M}_{g,n} \arrow[r, "\omega^{\otimes k}"] & \mathscr{J}ac_g^{k(2g-2)}. \end{tikzcd}$$

Though initially studied in Teichmüller dynamics, this space was recently brought to the attention of algebraic geometers. Among others, Kontsevich, Zorich, Eskin, H. Masur, Möller, McMullen, Mirzakhani, Veech from the dynamics side and Farkas, Grushevsky, Pandharipande, Pixton, Polishchuk, D. Chen, Zvonkine from the algebraic geometry side, have studied this object in different contexts for different k's and partitions. The most interesting case is when $k = 1$, as the space $\mathcal{H}_g(\mu)$ is closely related to moduli of flat surfaces. We will briefly explain the connection in the next section. Let's see some examples.

Example. When the partition is $\mu = (2, \dots, 2)$, then the condition

$$\sum_1^{g-1} 2x_i \sim K_C$$

is equivalent to $x_1 + \ldots + x_{g-1}$ being a theta characteristic, i.e. a square root of the canonical divisor. It is well known (cf. [Mum71]) that for a family $\pi : \mathscr{X} \to B$ over a connected base, if

$$\eta^{\otimes 2} \cong \omega_\pi,$$

the value $h^0(X_b, \eta_b) \mod 2$ is constant. Moreover, two spin curves $[C, \eta]$ and $[C', \eta']$ can be deformed to each other (i.e., there exists a family of spin curves over a connected base such that $[C, \eta]$ and $[C', \eta']$ appear as fibers) if and only if the parity of $h^0(\eta)$ and $h^0(\eta')$ is the same. In other words the moduli space of spin curves (curves with a theta characteristic) has two connected components, even and odd. There is a natural map

$$\begin{array}{rcl} \mathcal{H}_g(2, \dots, 2) & \to & \mathcal{S}_g^+ \coprod \mathcal{S}_g^- \\ [C, x_1, \dots, x_{g-1}] & \mapsto & [C, \mathcal{O}_C(x_1 + \ldots + x_{g-1})] \end{array}$$

that dominates $\mathcal{S}^- \coprod Z$, where Z is the divisor in the even component given by

$$Z := \left\{[C, \eta] \in \mathcal{S}_g^+ \mid h^0(\eta) \geqslant 2\right\} \subset \mathcal{S}_g^+.$$

From this, one already sees that $\mathcal{H}_g(\mu)$ is not always connected.

The space in not connected for a short list of partitions. Perhaps more surprising is the fact that $\mathcal{H}_g^k(\mu)$ is always smooth, regardless of the partition. Because of this, connected components and irreducible components coincide. This two statements will be treated in the coming subsections, §1.3.3 and §1.3.4.

Example. Another interesting example for a partition of the same length is

$$\mu = (g, 1, 1, \ldots, 1).$$

Recall that, a Weierstrass point $x \in C$, is defined by the condition

$$h^0(C, \mathcal{O}_C(gp)) \geqslant 2 \quad \text{or equivalently} \quad h^0(C, K_C(-gp)) \geqslant 1.$$

There is a map to the Weierstrass divisor

$$\begin{array}{rcl} \pi : \mathcal{H}_g(g, 1, \ldots, 1) & \to & \mathcal{W} \subset \mathcal{M}_{g,1} \\ [C, x_1, \ldots, x_{g-1}] & \mapsto & [C, x_1]\,. \end{array}$$

Moreover, for a general point $[C, p] \in \mathcal{W}$, the divisor $K_C - gp$ is effective of degree $g - 2$, therefore of the form

$$K_C - gp \sim x_1 + \ldots + x_{g-2}.$$

One can see that the map π factors through the symmetric quotient

$$\mathcal{H}_g(g, 1, \ldots, 1) \to \mathcal{H}_g(g, 1, \ldots, 1)/\Sigma_{g-2} \to \mathcal{W} \tag{1.1}$$

and the last map is a birational isomorphism, meaning, the map (1.1) is generically finite and dominant of degree $(g-2)!$.

From these two examples one can see that choosing different partitions we recover interesting loci in $\mathcal{M}_{g,n}$. Both, the Weierstrass locus and the moduli space of spin curves, are much studied objects. The strata seems to englobe different loci providing us with a more general picture. For instance, the space $\mathcal{S}_g^-$ has rationality properties for $g \leqslant 11$, more concretely, it can be covered by rational curves, cf. [FV14]. We will see in the last chapter (Theorem 4.2) that this is the case for every partition of length $g - 1$, moreover, for every partition of length at least $g - 1$. An important part of this thesis is concerned with the following problem:

Problem. Describe the geometry of $\mathcal{H}_g^k(\mu)$.

One of the most important invariants in complex geometry is the *Kodaira dimension* (denoted Kod), an invariant that measures the complexity of the variety in question (we will precisely define this in §1.4). At one extreme there are *uniruled varieties* (with Kod $= -\infty$). These are varieties having rational curves passing through a general point. At the other extreme (Kod=dimension of the variety), we have varieties of general type. In birational geometry, varieties that are almost isomorphic to a projective space are called *rational varieties*. One step higher in complexity are *unirational varieties*, varieties that are dominated by a projective space; this means that one can parametrize the variety using projective coordinates. The most complex class of varieties that lie still in the negative Kodaira dimension range are those that can be covered by rational curves, that is, uniruled varieties. We will give partial answer in small genus cases to the question:

Question. What is the Kodaira dimension of $\mathcal{H}^k_g(\mu)$?

Before going to the birational geometry of the strata, there are more fundamental aspects to be treated.

Question. Some of the most basic questions that we can ask about the strata are the following:

- What is the dimension of $\mathcal{H}^k_g(\mu)$?
- How many connected components it can have?
- Is $\mathcal{H}^k_g(\mu)$ singular? What is the tangent space over a point?

There are three different incarnations of the strata with different notations throughout the literature. The one that we will treat is $\mathcal{H}^k_g(\mu)$, but some of the results are given for the two other incarnations that we list in the following definition:

Definition 1.15. The k-Hodge bundle is defined to be the total space of the vector bundle $\pi_*\omega_\pi^{\otimes k}$, that is,

$$\mathbb{E}^k_g := \mathrm{Tot}(\pi_*\omega_\pi^{\otimes k}),$$

where $\pi : \mathscr{C} \to \mathcal{M}_g$ is the universal curve. For a partition μ of $k \cdot (2g-2)$ with positive entries, we define

$$\Omega^k_g(\mu) := \left\{(C,\omega) \in \mathbb{E}^k_g \mid \text{ there are points } x_i \text{ with } \mathrm{div}(\omega) = \sum m_i x_i\right\}.$$

Notice that the vanishing of ω is constant up to scalar multiplication of ω, motivating the following definition;

$$\mathcal{P}^k_g(\mu) := \mathbb{P}\Omega^k_g(\mu) = \left\{(C,[\omega]) \in \mathbb{P}\left(\mathbb{E}^k_g\right) \mid \mathrm{div}(\omega) = \sum m_i x_i\right\}.$$

When $k = 1$ we omit it in the notation. The spaces $\Omega^k_{g,n}(\mu)$ and $\mathcal{P}^k_{g,n}(\mu)$ are constructed in the same way, replacing π by the universal curve over $\mathcal{M}_{g,n}$ and the condition is

$$(C, x_1, \dots, x_n, \omega) \in \Omega^k_{g,n}(\mu) \text{ if and only if } \mathrm{div}(\omega) = \sum m_i x_i.$$

When the partition has negative entries, we follow the same construction for a twisted version of $\mathbb{E}_g^k$. Let μ be a partition with r-negative entries, of the form

$$\mu = (-\mu_1, \mu_2) = (-m_1, \ldots, -m_r, m_{r+1}, \ldots, m_n).$$

Let $\pi : \mathscr{C} \to \mathcal{M}_{g,n}$ be the universal curve over $\mathcal{M}_{g,n}$, with sections $s_1, \ldots, s_n : \mathcal{M}_{g,n} \to \mathscr{C}$ and associated divisors $D_i = s_{i*}[\mathcal{M}_{g,n}]$, we define the twisted k-Hodge bundle associated with μ as

$$\mathbb{E}_g^k(-\mu_1) := \mathrm{Tot}\left(\pi_* \omega_{\pi,-\mu_1}^{\otimes k}\right),$$

where $\omega_{\pi,-\mu_1}^{\otimes k}$ is the twisted line bundle given by

$$\omega_\pi^{\otimes k} \otimes \mathcal{O}_{\mathscr{C}}\left(\sum_1^r m_i D_i\right).$$

The strata is defined by

$$\Omega_g^k(\mu) := \left\{ (C, \omega) \in \mathbb{E}_g^k(-\mu_1) \mid \text{ there are points } x_i \text{ with } \mathrm{div}(\omega) = \sum_{r+1}^n m_i x_i \right\}$$

and its projectivization: $\mathcal{P}_g^k(\mu) := \mathbb{P}\Omega_g^k(\mu)$.

Some of the relations between $\mathcal{H}$, $\mathcal{P}$ and Ω are summarized in the following proposition. The proof is straightforward and we included it for sake of completeness.

Proposition 1.16. *Let*

$$\mu = (\underbrace{m_1, \ldots, m_1}_{n_1}, \underbrace{m_2, \ldots, m_2}_{n_2}, \ldots, \underbrace{m_r, \ldots, m_r}_{n_k})$$

be a partition of $k \cdot (2g-2)$, *where* $n_1 + \ldots + n_k = n$. *The product of symmetric groups*

$$\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$$

acts on $\mathcal{H}_g^k(\mu)$ *by permuting the points with the same vanishing and*

$$\mathcal{H}_g^k(\mu)/\Sigma_{n_1} \times \ldots \times \Sigma_{n_k} \cong \mathcal{P}_g^k(\mu).$$

By definition $\Omega_{g,n}^k(\mu)$ *is a* $\mathbb{C}^*$*-bundle over* $\mathcal{P}_{g,n}^k(\mu)$ *and*

$$\mathcal{H}_g^k(\mu) \cong \mathcal{P}_{g,n}^k(\mu).$$

Proof. It is a standard fact that, for two meromorphic sections of a line bundle $s, s' \in \Gamma(U, L)$ having the same vanishing $\operatorname{div}(s) = \operatorname{div}(s')$, there is a non-zero constant $\lambda \in \mathbb{C}$ such that $s = \lambda s'$. Therefore, $\mathcal{P}^k_g(\omega)$ can be read as the space of k-canonical divisors, that is, tuples $(C, \sum m_i x_i)$, where $\sum m_i x_i$ is the divisor associated to a meromorphic section of $\omega^{\otimes k}$. On the other hand the quotient is the space of tuples

$$[C, x_1 + \ldots + x_{n_1}, \ldots, x_{n-n_k+1} + \ldots + x_n] \in \mathcal{H}^k_g(\mu)/\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$$

such that

$$m_1 (x_1 + \ldots + x_{n_1}) + \ldots + m_r (x_{n-n_k+1} + \ldots + x_n) \sim kK_C.$$

The morphisms in both directions are the obvious ones. The last statement is proved in the same way. □

In the coming sections we will describe the connected components, following [KZ03] and sketch the proof for smoothness, following [Pol06]. First we treat the question about the dimension.

1.3.1 Dimension of the strata

The fist thing to notice is that, when the partition has positive entries, $\mathcal{H}^k_g(\mu)$ can be described as a degeneracy locus of certain vector bundle map over $\mathcal{M}_{g,n}$. As before μ is a partition of $k \cdot (2g-2)$ of length n and positive entries. Let

$$\pi : \mathscr{C} \to \mathcal{M}_{g,n}$$

be the universal curve with sections $s_1, \ldots, s_n : \mathcal{M}_{g,n} \to \mathscr{C}$ and $D_i = s_{i*}[\mathcal{M}_{g,n}]$. One can check that both sheaves,

$$\mathscr{E} := \pi_* \omega_\pi^{\otimes k} \quad \text{and} \quad \mathscr{F} := \pi_* \left(\omega_\pi^{\otimes k} \otimes_{\mathcal{O}_\mathscr{C}} \mathcal{O}_{m_1 D_1 + \ldots + m_n D_n} \right),$$

are vector bundles and the push forward of the evaluation map gives us the vector bundle map

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "\phi"] \arrow[dr, dash] & & \mathscr{F} \arrow[dl, dash] \\ & \mathcal{M}_{g,n}. & \end{tikzcd}$$

Over a point $[C, x_1, \ldots, x_n]$, the map is given by evaluation

$$\phi_{[C, x_1, \ldots, x_n]} : H^0(C, K_C^{\otimes k}) \to H^0 \left(C, K_C^{\otimes} |_{\sum m_i x_i}\right).$$

By Riemann-Roch, one can see that

$$\operatorname{rk}(\mathscr{F}) = k \cdot (2g-2) \quad \text{and} \quad \operatorname{rk}(\mathscr{E}) = \begin{cases} g, & \text{if } k = 1; \\ 2k(g-1) + 1 - g, & \text{if } k \geqslant 2. \end{cases}$$

A point $[C, x_1, \ldots, x_n] \in \mathcal{M}_{g,n}$ lies in the strata $\mathcal{H}_g^k(\mu)$ if and only if

$$\mathrm{rk}(\phi_{[C,x_1,\ldots,x_n]}) \leqslant \mathrm{rk}\, E - 1.$$

We find the following formula for the expected codimension as a degeneracy locus

$$\text{exp-codim}(\mathcal{H}_g^k(\mu)) = k(2g-2) - (\mathrm{rk}(\mathscr{E}) - 1) = \begin{cases} g-1, & \text{if } k = 1; \\ g, & \text{if } k \geqslant 2. \end{cases}$$

When the partition has negative entries

$$\mu = (-\mu_1, \mu_2) = (-m_1, \ldots, -m_r, m_{r+1}, \ldots, m_n).$$

We can compute the expected dimension replacing $\mathbb{E}_g^k$ with

$$\mathbb{E}_g^k(-\mu_1) := \pi_* \left(\omega_\pi \otimes \mathcal{O}_{\mathscr{C}}(m_1 D_1 + \ldots + m_r D_r)\right).$$

The vector bundle map over the point $[C, x_1, \ldots, x_n]$ is given by

$$H^0\left(C, \omega_C\left(\sum_1^r m_i x_i\right)\right) \to H^0\left(C, \omega_C\left(\sum_1^r m_j x_j\right) \otimes \mathcal{O}_{\sum_{r+1}^n m_j x_j}\right).$$

The locus where this fails to have maximal rank defines $\mathcal{H}_g^k(\mu)$, and by the same argument as before, the expected codimension is

$$\text{exp-codim}(\mathcal{H}_g^k(\mu)) = g.$$

The lack of dependency on k, n and μ is rather surprising. Moreover, if the expected codimension equals the codimension, one can use Porteous formula to compute the class of $\mathcal{H}_g^k(\mu)$ in the Chaw ring of the open $\mathcal{M}_{g,n}$.

There are essentially three ways to show that the expected dimension is indeed the dimension. From the general theory of determinantal varieties (see [ACGH85, Ch. II §4]) we obtain

$$\text{codim}\ (\mathcal{H}_g^k(\mu)) \leqslant \text{exp-codim}(\mathcal{H}_g^k(\mu)).$$

The first strategy is to show that there is a $2g-1$ (resp. $2g-2$) dimensional locus in $\mathcal{M}_{g,n}$ that is not contained in $\mathcal{H}_g(\mu)$ (respectively $\mathcal{H}_g^k(\mu)$). The second a third strategies rely on local study. This can be done using transcendental methods that allow us to, not only compute the dimension, but find local coordinated for $\mathcal{H}_g^k(\mu)$. We will treat the case when $k = 1$ and μ is holomorphic in the rest of this subsection. The dimension can also be computed using algebraic techniques, more explicitly, deformation theory. We will refer to the last approach in subsection §1.3.4.

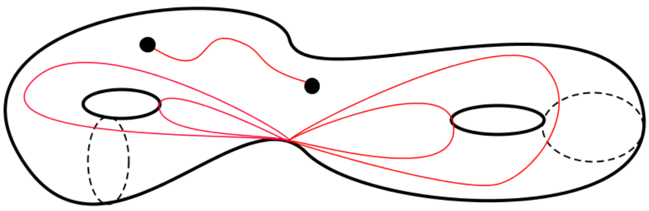

Figure 1.1: Basis of $H_1(C,\{x_1,x_2\};\mathbb{Z})$ for a genus 2 curve.

For an n-pointed curve $(C,x_1,\ldots,x_n)$, let $\gamma_1,\ldots,\gamma_{2g+n-1}$ be a base for the relative homology group $H_1(C,\{x_1,\ldots,x_n\};\mathbb{Z})$, where

$$\langle\gamma_1,\ldots,\gamma_{2g}\rangle = H_1(C,\mathbb{Z})$$

is the standard symplectic basis and γ_{2g+i} is the class of a cycle connecting x_1 and x_{i+1}, i.e.,

$$\partial\gamma_{2g+i} = [x_{i+1}] - [x_1], \quad \text{for } \; i = 1,\ldots,n-1.$$

We represent a point $(C,\omega)\in\Omega_g(\mu)$ in $\mathbb{C}^{2g+n-1}$ by the complex vector

$$(C,\omega)\mapsto\left(\int_{\gamma_1}\omega,\ldots,\int_{\gamma_{2g+n-1}}\omega\right)\in\mathbb{C}^{2g+n-1}.$$

Or equivalently, $\omega\in H^{1,0}(C)$ defines a point in

$$H^1(C,\{x_1,\ldots,x_n\};\mathbb{C})\cong H^1(C,\{x_1,\ldots,x_n\};\mathbb{Z})\otimes\mathbb{C}$$

by integration along the paths connecting the points. If $(C',\omega')\in\Omega_g(\mu)$ lies in an analytic neighborhood of (C,ω), with $\mathrm{div}(\omega') = \sum m_i x_i'$, one can identity

$$H^1(C,\{x_1,\ldots,x_n\};\mathbb{C})\cong H^1(C',\{x_1',\ldots,x_n'\};\mathbb{C}).$$

It is proved in [KZ96], that $H^1(C,\{x_1,\ldots,x_n\};\mathbb{C})$ provide us with a local chart of $\Omega_g(\mu)$ around the point (C,ω). We refer to the image point

$$(C,\omega)\mapsto H^1(C,\{x_1,\ldots,x_n\};\mathbb{C})\cong\mathbb{C}^{2g+n-1} \tag{1.2}$$

as *period coordinate* for (C,ω). From this follows that, when the partition has non-negative entries,

$$\dim\Omega_g(\mu) = 2g+n-1 \quad \text{and} \quad \dim\mathcal{P}_g(\mu) = 2g+n-2,$$

where n is the length of μ. Finally, by Proposition 1.16, $\dim\mathcal{H}_g(\mu) = 2g+n-2$.

The question now is what happens when we allow μ to have negative entries or when $k\geqslant 2$. We can still conclude that, the dimension is the expected one, by computing the dimension of the tangent space over a general point. This will be explained in the subsection about smoothness, §1.3.4.

1.3.2 Flat surfaces and Teichmüller dynamics

The goal of this section is to explain some of the main transcendental aspects of the strata $\mathcal{H}_g(\mu)$. We will skip most of the proofs and refer to [Wri15] or [Che17c], two wonderful surveys that flaunt a more detailed exposition.

An abelian differential ω on a Riemann surface C defines a flat structure such that C can be realized as a plane polygon with edges identified via translation. Varying the polygon by the action of $GL(2, \mathbb{R})$ induces an action on the moduli space of pairs (C, ω). The dynamics of this action is what is usually referred to as Teichmüller dynamics. Their orbit closures are known to be affine invariant submanifolds and they have shown to be extremely useful in understanding the geometry of the global object in question. There are still many open questions regarding this action and its orbits in the moduli space of curves.

Definition 1.17. A *flat surface* is a closed topological surface X together with a finite set of points $\Sigma \subset X$, such that

- there is an atlas of charts to $\mathbb{C}$ on $X \setminus \Sigma$ whose transition maps are translations and
- for each $p \in \Sigma$, there exists an integer $k > 0$ such that, a neighborhood of p is homeomorphic to the gluing of $2k + 2$ half discs with gluing scheme as in Figure 1.2 which is an isometry away from p.

Notice that $X \setminus \Sigma$ is endowed with the Euclidean metric from $\mathbb{C}$. We call a point $p \in \Sigma$ a *singularity* of the metric with *cone angle* $2\pi(k+1)$.

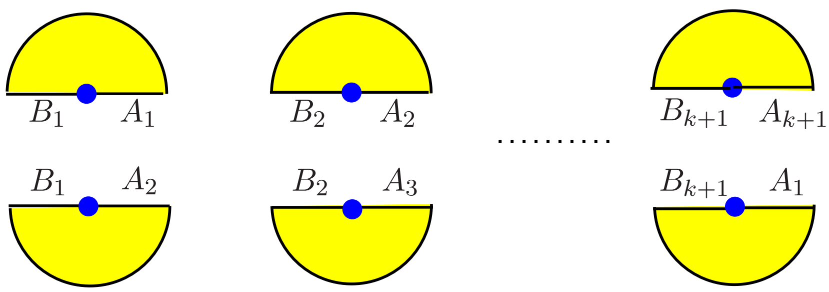

Figure 1.2: Neighborhood around $p \in \Sigma$. The figure was taken from [Che17c, p. 3].

Proposition 1.18. *Let μ be an holomorphic partition of $2g - 2$. Then any point $(C, \omega) \in \Omega_g(\mu)$ is a flat surface. On the other hand, any flat surface (X, Σ) induces a pair (C, ω), with ω vanishing at the singularities Σ with order k for those points having cone angle $2\pi(k+1)$.*

Proof. If ω vanishes with order k and $p \in C$, then we choose a local coordinate w such that $\omega = w^k g(w)dw$, with $g(0) \neq 0$. We change coordinates $w \mapsto z$, where

$$z^{k+1} := (k+1)\int_0^w g(t)t^k dt.$$

Notice that $z^k dz = \omega$. This shows that there are charts such that $\omega = dz$ away for the zeros and $\omega = z^k dz$ at the zeros of order k. Away from the zeroes, the choice is unique up to translation since $dz = d(z+c)$. The atlas induced by this charts forms a flat surface.
On the other hand, if we start with a flat surface (X, Σ), we define $\omega := dz$ away from Σ and at $p \in \Sigma$ with cone angle $2\pi(k+1)$, we define $\omega := z^k dz$, where z is the local chart at the neighboorhod of p. One can check that ω defined a global 1-form at X and $(X, \omega) \in \Omega_g(\mu)$. □

The name "flat" comes from the fact that, the charts of $X \setminus \Sigma = C \setminus \mathrm{Zeros}(\omega)$, locally induce the standard Euclidean metric on $X \setminus \Sigma$ and, since transition functions are translations, the metric is absent of curvature. The flat metric does not extend to the singularities Σ, that is why we use the term "singularity". This is an easy consequence of Gauss-Bonnet Theorem that, for a Riemann surface, relates the curvature K with the topological Euler characteristic:

$$\int_X K dx \wedge dy = 2\pi(2-2g).$$

A high genus curve must have non-vanishing curvature, i.e., it cannot be flat. At a singular point $p \in \Sigma$, the metric is the pull-back of the Euclidean metric on $\mathbb{C}$ by the map $z \mapsto z^{k+1}$, which is exactly the one of Figure 1.2. Let us illustrate this with an example.

Example 1.19. We start with a polygon as in Figure 1.3 and identify opposite edges. Notice that opposite edges differ by a translation. If Σ is the set of vertices represented by color points in the picture, then the transition charts of $X \setminus \Sigma$ are translations. For the fist polygon, there is one singular point (red point) of cone angle $6\pi = 2\pi(2+1)$ and for the second polygon, there are two singular points (red and blue point), each with cone angle 4π. Thus, both are Riemann surfaces of genus two and the abelian differential induced by the flat structure has vanishing

$$\mathrm{Zeroes}(\omega) = 2 \cdot \textcolor{red}{\bullet}$$

for the fist polygon and

$$\mathrm{Zeroes}(\omega) = 1 \cdot \textcolor{red}{\bullet} + 1 \cdot \textcolor{blue}{\bullet}$$

for the second one.

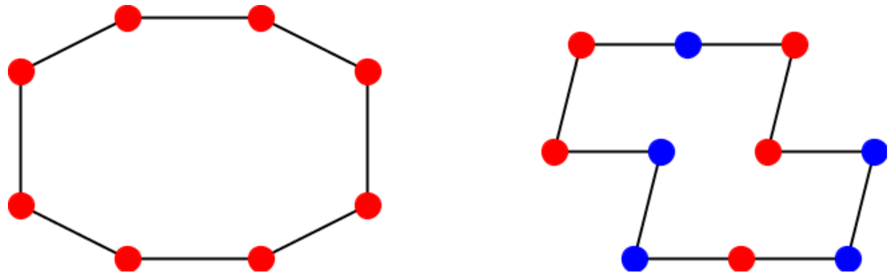

Figure 1.3: Example of flat surfaces of genus two in $\Omega_2(2)$ and $\Omega_2(1,1)$ respectively.

It turns out that every flat surface can be realized as a polygon in the plane with edges of te same length identified via translation. This is a statement that deserves a proof, we refer to [Wri15] for a detailed proof of this fact.

Let $GL^+(2,\mathbb{R})$ be the group of 2×2 matrices with positive determinant. This action has a natural orientation preserving action on $\mathbb{C}$, moreover, the conjugation of a translation is again a translations and the cone angle of a singular point is preserved. Therefore, it induces an action on the moduli of flat surfaces $\Omega_g(\mu)$. See Figure 1.4.

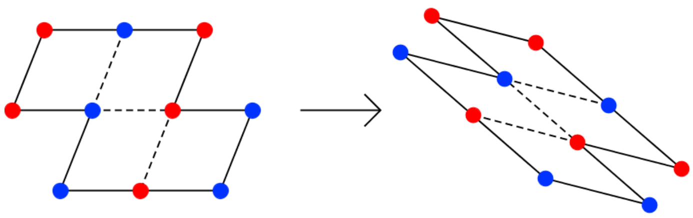

Figure 1.4: Action of $GL^+(2,\mathbb{R})$ on flat surfaces.

The understanding of the $GL^+(2,\mathbb{R})$-orbits closures in $\Omega_g(\mu)$ is one of the central tasks in Teichmüller dynamics. We discuss two fundamental results in this regard. Masur and Veech in [Mas82] and [Vee82] showed that for almost all $(C,\omega) \in \Omega_g(\mu)$, the closure of the $GL^+(2,\mathbb{R})$-orbit of (C,ω) is the whole stratum (or a connected component of it). However, for special points in the stratum, their orbit closure can be a proper subset. The second fundamental result regarding the nature of orbit closures is more recent, due to Eskin, Mirzakhani and Mohammadi in [EMM15]. The showed that every orbit closure is an *affine invariant submanifold* in $\Omega_g(\mu)$, meaning that locally is given by the vanishing of real linear homogenous equations in the period coordinates 1.2. Moreover they are algebraic over $\overline{\mathbb{Q}}$, cf. [Fil16].

Of special interest are the orbit closures whose image in $\mathcal{M}_g$, under the forgetful map $(C,\omega) \mapsto [C]$, are of complex dimension one. They are called

Teichmüller curves and they have a long list of surprising properties, see [Che17c, §3]. They where used in [Che10] to estimate the growth of the slope of $\mathcal{M}_g$. See [CFM13, §3.5] for a comprehensive survey on the matter.

1.3.3 Connected components of the strata

Recall that the space $\mathcal{H}_g(2,\dots,2)$ must break into at least two connected components. If we consider the symmetric quotient

$$\mathcal{P}_g(2,\dots,2) = \mathcal{H}_g(2,\dots,2)/\Sigma_{g-1},$$

the forgetful map dominates

$$\mathcal{S}_g^- \coprod Z.$$

Moreover, for a general odd spin curve $[C,\eta] \in \mathcal{S}^-$, one has $h^0(\eta) = 1$. This means, there is a unique divisor $\sum x_1$, with $\eta = \mathcal{O}_C\left(\sum x_i\right)$. On the other hand for a general $[C,\eta] \in Z$, one has $h^0(\eta) = 2$. Thus, the map

$$\mathcal{P}_g(2,\dots,2) \to \mathcal{S}^- \coprod Z$$

is birational over $\mathcal{S}^-$ and birationally a $\mathbb{P}^1$-bundle over Z. The locus Z is a divisor in $\mathcal{S}^+$, known in the literature (e.g. [Far10b]) as the *theta null* divisor and usually denoted by

$$\Theta_{null} := \left\{[C,\eta] \in \mathcal{S}^+ \mid H^0(C,\eta) \neq 0\right\}.$$

Remark 1.20. The divisor Θ_{null} has a strong resemblance with the classical theta divisor in the Jacobian of a curve. In the latter case, it is easy to see that consist of an irreducible divisor, since it is isomorphic to the image of the Abel-Jacobi map

$$C^{\times g-1} \to \mathrm{Jac}^{g-1}(C).$$

In [Tei88], Teixidor i Bigas showed that the locus of curves in $\mathcal{M}_g$ that admit a theta characteristic with at least two sections is an irreducible divisor. Probably, by looking at the monodromy of the finite map $\mathcal{S}_g^+ \to \mathcal{M}_g$, the irreducibility of Θ_{null} can be established. In any case, as simple consequence of smoothness of the strata and the complete account of its possible connected components, one can easily deduce the irreducibility of Θ_{null}.

From the analysis above, it follows that $\mathcal{P}_g(\mu)$ has at least two connected components corresponding to the parity of the associated spin structure. Recall that for a partition of $2g-2$,

$$\mu = (\underbrace{m_1,\dots,m_1}_{n_1},\underbrace{m_2,\dots,m_2}_{n_2},\dots,\underbrace{m_r,\dots,m_r}_{n_k}),$$

the quotient

$$\mathcal{H}_g(\mu) \to \mathcal{H}_g(\mu)/\left(\Sigma_{n_1} \times \dots \times \Sigma_{n_k}\right).$$

is isomorphic to $\mathcal{P}_g$ and $\Omega_g(\mu) \to \mathcal{P}_g$ is a $\mathbb{P}^1$-bundle, cf. Proposition 1.16. By the following lemma, the account of connected components for $\Omega_g(\mu)$ carries on to $\mathcal{H}_g(\mu)$.

Lemma 1.21. *The monodromy action on the general fibers of the finite map*

$$\mathcal{H}_g^k(\mu) \to \mathcal{P}_g^k(\mu)$$

is transitive.

Proof. This lemma is a simple consequence of a local surgery called *breaking up a zero* which modifies the curve in a small neighborhood of a chosen zero of a differential, see [KZ03, §4.2]. Let us assume $k = 1$ and $[C, x_1, \dots, x_n] \in \mathcal{H}_g(\mu)$ with x_i and x_j close enough so that C has a neighborhood around x_i and x_j coming from breaking up a zero of order $m_i + m_j$. See Figure 1.5. By rotating the neighborhood an angle of $\pi \cdot (m_i + m_j + 1)$, we interchange x_i with x_j. See also [KZ03, Lemma 9 and 10]. For $k \geqslant 2$, we use the *canonical cover construction*, cf. [BCGGM16b, §4]. For a k-th differential η on a curve C, there is a unique cyclic cover $\pi : \widehat{C} \to C$ such that $\pi^*\eta = \omega^{\otimes k}$ is the k-th power of an abelian differential ω. The abelian differential is defined up to a k-th root of unity, but the canonical divisor $\mathrm{div}(\omega)$ is well defined, independent of the choice of root of unity. This gives us an embedding $\mathcal{P}_g^k(\mu) \hookrightarrow \mathcal{P}_g(\hat{\mu})$ and the argument reduces to the $k = 1$ case. $\square$

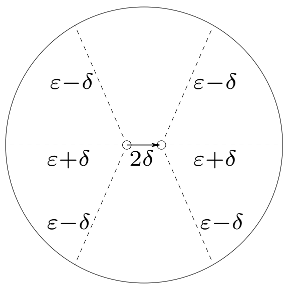

Figure 1.5: Example of a neighborhood after breaking up a zero of cone angle 6π. The total cone angle around each of the two singular points is 4π.

This lemma implies that, the space $\mathcal{H}_g(\mu)$ has the same number of connected components as $\mathcal{P}_g(\mu)$ and $\Omega_g(\mu)$. This is pointed out as a remark by Kontsevich and Zorich, cf. [KZ03, Remark 1]. The full list of possible connected components of $\Omega_g(\mu)$ (and therefore of $\mathcal{H}_g(\mu)$) is described in the following list.

Theorem 1.22 (Theorems 1 and 2 in [KZ03]). *For $g \geqslant 4$, the connected components of $\Omega_g(\mu)$ and therefore the connected components of $\mathcal{H}_g(\mu)$ are described in the flowing list:*

- *When* $\mu = (2g-2)$, *the space* $\mathcal{H}_g(\mu)$ *breaks into three connected components; the hyperelliptic one* $\mathcal{H}_g^{hyp}(\mu)$ *and the two other corresponding to curves with even and odd associated spin structure* $\mathcal{H}_g^+(\mu)$ *and* $\mathcal{H}_g^-(\mu)$.
- *When* $\mu = (2l, 2l)$ *the space* $\mathcal{H}_g(\mu)$ *breaks into three connected components; hyperelliptic, even and odd. We also denoted by* $\mathcal{H}_g^{hyp}(\mu)$, $\mathcal{H}_g^+(\mu)$ *and* $\mathcal{H}_g^-(\mu)$.
- *When* $\mu = (2l_1, \dots, 2l_n)$ *different from the two above, there are two connected components; even and odd.*
- *When* $\mu = (2l-1, 2l-1)$, *there are two connected components; the hyperelliptic one* $\mathcal{H}_g^{hyp}(\mu)$ *and* $\mathcal{H}_g^{non-hyp}(\mu)$.
- *For every other partition* μ, *the strata is non-empty and connected.*

Finally for $g = 1, 2$ *and* 3, *the connected components of the strata are described in the following list:*

- *The space* $\mathcal{H}_1(0)$ *is connected, equals to* $\mathcal{M}_{1,1}$.
- *Both* $\mathcal{H}_2(1,1)$ *and* $\mathcal{H}_2(2)$ *are connected.*
- *Finally* $\mathcal{H}_3(2,2)$ *and* $\mathcal{H}_3(4)$ *have two connected components corresponding to hyperelliptic and non-hyperelliptic. The last one* $\mathcal{H}_3(1,3)$ *is connected.*

1.3.4 Smoothness of strata

Smoothness of the strata of holomorphic differentials, i.e., for holomorphic partitions with $k = 1$, was obtained by Polishchuk in [Pol06]. The generalization to any partition with $k \geqslant 1$ was obtained simultaneously by Schmitt, [Sch16] and the five authors [BCGGM16b, Thm. 1.1]. We will give an account of the results and explain the main ideas in [Pol06] and [Sch16].

Recall the cartesian diagram

$$\begin{tikzcd} \mathcal{H}_g^k(\mu) \arrow[r] \arrow[d] & \mathcal{M}_{g,n} \arrow[d, "A_\mu"] \\ \mathcal{M}_g \arrow[r, "\phi_k"] & \mathscr{J}ac_g^{k(2g-2)}, \end{tikzcd}$$

where A_μ is the Abel-Jabobi map associated to μ and ϕ_k is the k-th canonical section

$$\phi_k : [C] \mapsto \left(C, \omega^{\otimes k}\right).$$

Both maps are sections, meaning, the composition with the natural projection is the identity on $\mathcal{M}_{g,n}$. For any point $[C, x] \in \mathcal{H}_g^k(\mu)$, from the diagram above, one concludes the tangent space of the strata at the point $[C, x]$ is given by the kernel of the map

$$d\phi_k - dA_\mu : T_{[C,x]}\mathcal{M}_{g,n} \oplus T_{[C,x]}\mathcal{M}_{g,n} \to T_{[C,x,\omega^{\otimes k}]}\mathcal{J}ac_g^{k\cdot(2g-2)}. \tag{1.3}$$

When $k \geqslant 2$, we will call μ *primitive* if not all entries are divisible by k, as before, *holomorphic* if all entries are non-negative and *meromorphic* otherwise. If μ is not primitive, then

$$\mathcal{O}_C\left(\sum m_i x_i\right) \cong \omega_C^{\otimes k} \quad \text{if and only if} \quad \mathcal{O}_C\left(\frac{1}{k}\sum m_i x_i\right) \cong \omega_C.$$

When the partition is not primitive,

$$\mathcal{H}_g^k(\mu) \cong \mathcal{H}_g\left(\frac{1}{k}\mu\right)$$

and the study reduces to the case $k = 1$.

Theorem 1.23 (Thm 1.1.a in [Pol06] , Thm. 1.1 and Prop. 2.1 in [Sch16])**.** *Let* μ *be a partition of* $k \cdot (2g-2)$*, primitive when* $k \geqslant 2$*. If* $k = 1$ *and* μ *is holomorphic, for any point* $[C, x] \in \mathcal{H}_g^k(\mu)$*,*

$$\dim \mathcal{H}_g^k(\mu) = \dim T_{[C,x]}\mathcal{H}_g^k(\mu) = 2g - 2 + n$$

and

$$\dim \mathcal{H}_g^k(\mu) = \dim T_{[C,x]}\mathcal{H}_g^k(\mu) = 2g - 3 + n$$

otherwise.

The tangent spaces of $\mathcal{M}_{g,n}$ and $\mathcal{J}ac$ are given by

$$T_{[C,x]}\mathcal{M}_{g,n} \cong H^1\left(C, T_C\left(-\sum x_i\right)\right) \quad \text{and} \quad T_{[C,x,L]}\mathcal{J}ac_g^{k\cdot(2g-2)} \cong H^1(C, A_{L,x}),$$

where $A_{L,x}$ is the Atiyah algebra of first order differential operators on L with symbol vanishing at the marked points $x_1, \ldots, x_n$, cf. [ACG11, Ch. XI §2 Thm 2.12] and [Dia84]. More explicitely, A_L is a rank two vector bundle sitting in the exact sequence

$$0 \to \mathcal{O}_C \to A_L \to T_C \to 0$$

and on an open U of C with coordinate z, $A_L(U)$ is generated by 1 and D where D is the operator that takes derivatives of sections of L with respect to z. The map on the left of the exact sequence is $D \mapsto \frac{\partial}{\partial z}$. Now $A_{L,x}$ sits in in the following exact sequence

$$0 \to \mathcal{O}_C \to A_{L,x} \to T_C(-x_1 - \ldots - x_n) \to 0,$$

and the local generators are $\langle 1, D\rangle$ if the open U does not contain x_i and $\langle 1, zD\rangle$ if it does and z is a local coordinate vanishing at x_i.

We fix d to be the dimension of the cokernel of the map (1.3),

$$d := \dim \operatorname{coker}(d\phi_k - dA_\mu).$$

The main claim is

$$\dim T_{[C,x]}\mathcal{H}_g^k(\mu) = 2g - 3 + n + d,$$

where $d = 0$ if $k = 1$ and μ is holomorphic and $d = 0$ otherwise. In [Pol06, Lemma 2.3] and [Sch16, Cor. 2.5], the induced map of tangent spaces

$$d\phi_k - dA_\mu : H^1\left(C, T_C(-x_1 \ldots - x_n)\right) \to H^1\left(C, A_{\omega_C^{\otimes k}, x}\right)$$

is explicitly computed, as well as the kernel and cokernel.

1.4 Some basics on birational geometry

One of the central questions in algebraic geometry is to classify varieties up to birational isomorphisms. This task involves the understanding of birational invariants, the study of different birational models for a given variety, the comparison of singularities and positivity properties of the canonical class between different birational models, among others. The understanding of the birational geometry of moduli spaces is a central question in modern algebraic geometry. Even for curves, many questions about the geometry of $\mathcal{M}_g$ remain open.

1.4.1 Rationality

We recall some basic definitions.

Definition 1.24. A rational map between algebraic varieties, $\phi : X \dashrightarrow Y$, is said to be a *birational map* if there are non-empty open sets $U_X \subset X, U_Y \subset Y$ such that ϕ is defined over U_X, its image is U_Y and the restriction

$$\phi_{U_X} : U_X \to U_Y$$

is an isomorphism. When such map exists, we say that X and Y are *birational* varieties.

Definition 1.25. A variety X is said to be

- *rational* if it is birational to a projective space,

- *unirational* if it is dominated by a rational variety, i.e., if there is a rational map $\phi : X' \dashrightarrow X$ and a non-empty open set $U \subset X$ contained in the image of ϕ, such that X' is rational. Finally, X is said to be
- *uniruled* if there is a variety Y and a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$ that does not factors through the projection to Y.

Observe that the list of implications is,

$$\text{rational} \Longrightarrow \text{unirational} \Longrightarrow \text{uniruled}.$$

There are two other fundamental intermediate rationality notions. These are *stably rational* and *rationally connected*. One of the main open questions in birational geometry is to determine which implications are strict and which ones are actually equivalences. This is usually referred to as the *Lüroth problem*. For instance, it is well known that, unirationality and rationality are equivalent notions among smooth surfaces, this is a consequence of Castelnuovo's rationality criterion stating that a smooth surface S is rational if and only if its first Betti number $b_1(X)$ and its bi-genus $h^0(S, K_S^{\otimes 2})$ vanish. In dimension three there are examples of unirational but not rational varieties, cf. [CG72], [IM71], [AM72]. To construct uniruled varieties that are not rational or even unirational is not hard; a projective bundles over a basis of high Kodaira dimension provides us with an example. To this day, we don't know a single example of a rationally connected variety that is not unirational. We refer to [KSC04] for a comprehensive introduction to the subject.

Proposition 1.26. *Any projective bundle over a rational variety is rational.*

Proof. Let $\pi : \mathscr{P} \to X$ be a projective bundle over a rational variety. For a small enough open $U \subset X$, the projection map π is isomorphic to

$$U \times \mathbb{P}^n \to U$$

and we may assume $U \subset \mathbb{P}^m$. It is enough to prove that $\mathbb{P}^m \times \mathbb{P}^n$ is rational, but this is clear since $\mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$. ☐

Observe that, two smooth varieties X and Y are birational if and only if their function fields (fields of meromorphic functions) $K(X)$ and $K(Y)$ are isomorphic. In particular, X is rational if and only if its function field is purely transcendental

$$K(X) \cong \mathbb{C}(z_1, \dots, z_n) := \mathrm{Frac}\left(\mathbb{C}\left[z_1, \dots, z_n\right]\right).$$

Similarly, X is unirational if there exists a field extension $K(X) \hookrightarrow \mathbb{C}(z_1, \dots, z_n)$, into a purely transcendental field, and X is uniruled if there exists an inclusion of the form $K(X) \hookrightarrow \mathbb{K}(t)$, where $\mathbb{K}$ is a transcendental extension of $\mathbb{C}$ of the form $K(Y)$ for Y a variety of dimension $\dim(x) - 1$ and $\mathbb{K} \subset \mathbb{K}(t)$ is a purely transcendental extension of transcendence degree one.

1.4.2 Kodaira dimension

Let L be a line bundle on a normal variety X. If, for each $m \geqslant 1$, the vector space global sections is non-zero, $h^0(X, L^{\otimes m}) \neq 0$, then, the linear system $|L^{\otimes m}|$ induces a rational map

$$X \dashrightarrow \mathbb{P}^{h^0(L^{\otimes m})-1}.$$

This map is defined outside the base locus of the linear system $\mathrm{Bs}|L^{\otimes m}|$ that, for m big enough, is never the whole variety. We denote by $\phi_{L,m}(X)$, the closure of the image of this map inside $\mathbb{P}^{h^0(L^{\otimes m})-1}$.

Definition 1.27. The *Iitaka dimension* of L is defined to be

$$\kappa(X, L) := \max\{\dim \phi_{L,m}(X) \mid m \geqslant 1\}$$

if $h^0(X, L^{\otimes m}) \neq 0$ for some $m \geqslant 1$ and $-\infty$ otherwise. Furthermore, if the variety is smooth, its *Kodaira dimension* is defined to be the Iitaka dimension of the canonical class

$$\mathrm{Kod}(X) := \kappa(X, K_X).$$

A projective smooth variety is said to be of *general type* if $\mathrm{Kod}(X) = \dim X$.

One can see from the definition that $\mathrm{Kod}(X) \in \{-\infty, 0, \ldots, \dim(X)\}$ and, if K_X is ample then the variety X is of general type. On the other hand it is not hard to check that for X and Y smooth projective varieties

$$\mathrm{Kod}(X \times Y) = \mathrm{Kod}(X) + \mathrm{Kod}(Y).$$

Using this, one can construct varieties of any allowed Kodaira dimension. For instance in dimension 3, with E and C smooth curves of genus 1 and $g \geqslant 2$ respectively,

$$\begin{array}{ll} \mathrm{Kod}(\mathbb{P}^3) = -\infty, & \mathrm{Kod}(E \times C \times C) = 2 \\ \mathrm{Kod}(E \times E \times E) = 0, & \quad\quad \text{and} \\ \mathrm{Kod}(E \times E \times C) = 1, & \mathrm{Kod}(C \times C \times C) = 3. \end{array}$$

An alternative definition for the Iitaka (and Kodaira) dimension is the following. For an effective line bundle L on a variety X, one can define the *graded ring of sections*

$$R(X, L) := \bigoplus_{d \geqslant 0} H^0(X, L^{\otimes d}).$$

One can check that $R(X, L) \cong R(X, L^{\otimes N})$ as graded rings and, if $R(X, L)$ is finitely generated,

$$\kappa(X, L) = \dim \mathrm{Proj}\, R(X, L).$$

The ring $R(X, L)$ is not always finitely generated and one of the recent breakthroughs in birational geometry, due to Birkar, Cascini, Hacon and McKernan

[BCHM10], ensures that the ring $R(X, K_X)$ is always finitely generated when X is of general type. In any case, when finite generation does not holds, one can always interpret κ as the transcendence of the fraction field $\mathrm{Frac}(R(X, L))$ minus one.

Let $f : \mathbb{P}^1 \to X$ be a non-trivial map. Every vector bundle over $\mathbb{P}^1$ splits. In particular

$$f^*T_X = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n),$$

with $a_1 \geqslant \ldots \geqslant a_n$. A rational curve f is called *free* is $a_n \geqslant 0$, that is, if f^*T_X is globally generated. On the other hand from the exact sequence

$$0 \to T_{\mathbb{P}^1} \to f^*T_X \to N_f \to 0$$

one deduce that $a_1 \geqslant 2$ and $\deg(f^*T_X) = -(K_X \cdot f(\mathbb{P}^1))$. If f is a free curve and $\sigma \in H^0(X, mK_X)$ is a global section, then σ must vanish on $f(\mathbb{P}^1)$. It is not hard to see that, over $\mathbb{C}$, uniruledness is equivalent to the existence of a covering family of free rational curves, cf. [Deb01, §4.2]. In particular, $h^0(mK_X) = 0$. Thus,

$$X \text{ uniruled} \Longrightarrow \mathrm{Kod}(X) = -\infty.$$

The opposite implication is still conjectural. One of the major recent achievements in the area consist of a partial result towards this conjecture due to Boucksom, Demailly, Paun and Peternell [BDPP13], where they separate the conjecture in two partial implications and proved one of them. They show that if K_X is not psudo-effective then X is uniruled.

1.4.3 Leray spectral sequence

Let $f : X \to Y$ be a morphism of varieties and $\mathscr{F}$ a coherent sheaf on X. There is a standard way to relate the cohomology of $\mathscr{F}$ on X and the cohomology of $Rf_*\mathscr{F}$ on Y, called *Leray spectral sequence*. We denote the Leray spectral sequence and its convergency in page two by

$$E_2^{p,q} = H^p(Y, R^q f_*\mathscr{F}) \Longrightarrow H^{p+q}(X, \mathscr{F}).$$

This notation carries a considerable amount of information, some of it being:

- Maps
$$d_2^{p,q} : H^p(Y, R^q f_*\mathscr{F}) \to H^{p+2}(Y, R^{q-1} f_*\mathscr{F}),$$
called *differentials of the spectral sequence in page two*, where R^{-1} is declared to be zero.

- The maps $d_2^{p,q}$ for a complex with cohomology $E_\infty^{p,q}$ (i.e., the spectral sequence converges in page two) and

- There is a filtration F of the graded object $H^\bullet(X, \mathscr{F})$, i.e., a filtration for every H^n such that

$$\mathrm{gr}_p := F_p H^{p+q} / F_{p-1} H^{p+q}$$

is isomorphic to $E_\infty^{p,q}$.

A particular instance is when the sequence degenerates, meaning, the differentials are all trivial. One example is when the higher derived images $R^j f_* \mathscr{F}$ vanish for $j > 0$. In this case, the existence of the filtration and convergency traduces in the following lemma that we will use in several occasions. See also [Laz04, Prop. B.1.1].

Lemma 1.28. *If all higher derived images vanish, i.e.,* $R^j f_* \mathscr{F} = 0$ *for* $j \geqslant 1$, *then*

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_* \mathscr{F}) \quad \textit{for all } i.$$

1.4.4 Birational invariants

For any morphism $f : X \to Y$ and line bundle L on Y,

$$H^0(X, f^* L^{\otimes m}) \cong H^0(Y, f_* f^* L^{\otimes m})$$

and by the projection formula $f_* f^* L^{\otimes m} \cong f_* \mathcal{O}_X \otimes L^{\otimes m}$. The condition

$$f_* \mathcal{O}_X \cong \mathcal{O}_Y$$

is satisfied when f is a *fibered space*, this means, the map f is surjective and the field $K(Y)$ is algebraically closed inside $K(X)$. In particular if we assume f surjective and Y normal, these conditions are satisfied. As a consequence, one has

$$\kappa(X, f^* L) = \kappa(Y, L).$$

Almost never the pull back of the canonical class, by a birational morphism, is again the canonical class. There are several ways to show the Kodaira is a birational invariant. The choice is a matter of taste, here we will deduce it as a consequence of a highly non-trivial, but well-know fact stated in the following two theorems. Both theorems follow from *Hironaka principalization theorem*, cf. [Hir64] and [Wlo05] for a survey on the matter. We will make use of these results in the coming chapters.

Theorem 1.29. *Let* $f : X \to Y$ *be a birational map between smooth projective varieties. Then,*

$$R^i f_* \mathcal{O}_X = 0 \quad \textit{for all } i \geqslant 1.$$

This is the property that characterize *rational singularities*. In other words, smooth varieties have rational singularities. This theorem is one of the consequences of Hironaka's work on resolution of singularities, in this case, the fact that the map f can be dominated by a sequence of blow-ups over smooth centers. See [Hir64, Ch. 0 §5]

Theorem 1.30. *Let $f : X \dashrightarrow Y$ be a birational morphism between smooth projective varieties. Then there exist a smooth variety $\widetilde{X}$ and morphisms*

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\sigma"'] \arrow[dr] & \\ X \arrow[rr, dashed, "f"] & & Y, \end{tikzcd}$$

such that σ is a sequence of blow-ups over X at smooth centers.

We will take these theorems for granted. Notice, the crucial point is that the resolution consist of blow ups at smooth centers. One can easily resolve f by restricting the projection $X \times Y \to Y$ to the Zariski closure of the graph of f, moreover this is a blow-up, but not necessarily over smooth centers.

Before coming back to the Kodaira dimension, we state some easy consequences of the first theorem, that will also be used to a great extent.

Corollary 1.31. *The integers $h^i(X, \mathcal{O}_X)$, $h^i(X, \omega_X)$ and $h^0(X, \Omega^i_X)$ are birational invariants of smooth projective varieties.*

Proof. The first one follows from Lemma 1.28 together with Theorem 1.29. The second one follows from Serre duality and the third one from the symmetry in the Hodge diamond, $h^{0,i}(X) = h^{i,0}(X)$. □

Corollary 1.32. *If $f : X \to Y$ is a birational morphism between smooth projective varieties, then $f_*\omega_X \cong \omega_Y$ and $R^i\omega_X = 0$ for $i \geqslant 1$.*

Proof. By relative Serre duality in the derived category of $D(Y)$,

$$Rf_*\omega_X \cong R\mathscr{H}om\,(Rf_*\mathcal{O}_X, \omega_Y)\,.$$

Since $Rf_*\mathcal{O}_X = \mathcal{O}_Y$, one has

$$R^if_*\omega_X \cong \mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y)$$

and one can see that $f_*\omega_X \cong \omega_Y$ and $R^if_*\omega_X = 0$ for $i \geqslant 1$. □

Regarding the Kodaira dimension, we have:

Proposition 1.33. *The Kodaira dimension is a birational invariant between smooth projective varieties.*

Proof. By the theorem on resolution of rational maps 1.30, it is enough to show that, for X and $Y \subset X$ smooth with Y of codimension at least two, and $\varepsilon : \widetilde{X} \to X$ the blow up of X along Y,

$$\mathrm{Kod}(\widetilde{X}) = \mathrm{Kod}(X).$$

If $\sigma \in H^0(\widetilde{X}, mK_{\widetilde{X}})$, then $\varepsilon_*\sigma$ is a rational form on X with poles concentrated in Y, but the codimension of Y in X is at least two, therefore, $\varepsilon_*\sigma$ is regular. Moreover, by smoothness assumptions E is ruled and such global forms must vanish at E. This proves that the push forward induces and inclusion $H^0(mK_{\widetilde{X}}) \subset H^0(mK_X)$. On the other hand, again by smoothness assumptions, one can check that $K_{\widetilde{X}} = \varepsilon^*K_X + (c-1)E$, where E is the exceptional divisor of ε and c is the codimension of $Y \subset X$. Thus, if $\eta \in H^0(mK_X)$, then $\varepsilon^*\eta \in H^0(m\varepsilon^*K_X)$, but

$$H^0(m\varepsilon^*K_X) \subset H^0(m(\varepsilon^*K_X + (c-1)E)) = H^0(mK_{\widetilde{X}}).$$

this proves the opposite inclusion. □

We add one more classical result without a proof.

Lemma 1.34 (Easy addition formula. See, e.g., §10 in [Iit81]). *Let $f : X \to Y$ be a fiber space between normal projective varieties, F a general fiber and L a line bundle on X then*

$$\kappa(X, L) \leqslant \dim Y + \kappa(F, L|_F).$$

Easy addition formula usually refers to the statement above, but for the Kodaira dimension. One can check that $\omega_F = \omega_X|_F$ and therefore,

$$\mathrm{Kod}(X) \leqslant \dim(Y) + \mathrm{Kod}(F).$$

The Kodaira dimension is not always additive, for instance, there are K3 surfaces that can be realized as elliptic fibrations over $\mathbb{P}^1$. In such case

$$\mathrm{Kod}(\mathbb{P}^1) + \mathrm{Kod}(F) < \mathrm{Kod}(S) < \dim(\mathbb{P}^1) + \mathrm{Kod}(F).$$

Subadditivity of Kod is best known as the *Iitaka $C_{n,m}$ conjecture* and proved in many cases. We summarize some of the known cases in the following theorem:

Theorem 1.35 ([Kaw79], [Kaw81], [Kol87], [Vie77], [Vie83], [CH11], [Fuj13] and [Bir09]). *Let $f : X \to Y$ be a fibration between smooth projective varieties and let F be a general fiber. The inequality*

$$\mathrm{Kod}(F) + \mathrm{Kod}(Y) \leqslant \mathrm{Kod}(X)$$

holds if

- $\dim F = 1$ *or* 2,
- *either F or Y is of general type,*
- *either F or Y has maximal Albanese dimension (e.g., if one of them is an abelian variety) or, if*
- $\dim(X) \leqslant 6$.

1.5 State of the art and further questions

The study of the birational geometry of moduli spaces is a very active area of research and many fundamental questions remain open. For curves, it is known that $\overline{\mathcal{M}}_g$ has rationality properties for $g \leqslant 16$ and it is of general type for $g \geqslant 24$. We refer to [Far10a] for a survey on the subject. The list of contributions and partial results in the matter is quite long and papers like [Seg30] and many others where also essential in the understanding of the birational geometry of $\overline{\mathcal{M}}_g$. We summarize the main known results regarding the birational geometry of $\mathcal{M}_g$ as follows:

- $\overline{\mathcal{M}}_g$ is unirational for $g \leqslant 10$, cf. [Sev15].
- $\overline{\mathcal{M}}_g$ is unirational for $g \leqslant 14$, cf. [Ser81], [Ver05], [CR84].
- $\mathcal{M}_{15}$ is rationally connected and $\mathcal{M}_{16}$ is uniruled, cf. [CR86], [CR91], [BV05].
- $\overline{\mathcal{M}}_g$ is of general type for $g \geqslant 24$ and $\mathrm{Kod}(\overline{\mathcal{M}}_{23}) \geqslant 1$, cf. [HM82], [Har84], [EH87].
- $\overline{\mathcal{M}}_{22}$ is of general type and $\mathrm{Kod}(\overline{\mathcal{M}}_{23}) \geqslant 2$, cf. [Far00].

The transition from negative to positive Kodaira dimension happens in the range $g \in \{17, 18, 19, 20, 21, 23\}$, but it is unclear where. Some of the questions to ask are whether for some g in this range, the moduli space is of intermediate type, i.e., is has Kodaira dimension between zero and $3g-2$. Another question would be if the Kodaira dimension grows monotonously of it can decrease. One expects, when the genus increases, the complexity of $\mathcal{M}_g$ increases with it. This intuition is confirmed by most of the known cases, but there is one example that suggest that it might be not always the case. As stated in the next section:

$$\mathrm{Kod}(\overline{\mathcal{M}}_{10,11}) \geqslant \mathrm{Kod}(\overline{\mathcal{M}}_{11,11}).$$

1.5.1 Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$

Following [HM82], Logan in [Log03] initiated the study of the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$. The moduli space $\mathcal{M}_{g,n}$ appears naturally in the structure of the boundary of $\overline{\mathcal{M}}_g$ and the collection of rationality results for small g and n is quite extensive. Logan pointed out [Log03, Thm 2.4] that given $g > 3$, the moduli $\overline{\mathcal{M}}_{g,n}$ is of general type for, all but finitely many n's. One deduces from Theorem 1.35 that for fixed genus $g > 3$, the numerical function

$$k(n) := \mathrm{Kod}(\overline{\mathcal{M}}_{g,n})$$

is strictly increasing, as soon as $k(n) \geqslant 0$. Moreover, if $k(n) \geqslant 0$, the next one, $k(n+1)$, is at least $3g-2$. See the proof of Corollary 3.13 in section §3.4. Following notation in the literature, consider the numerical functions:

$$f(g) := \max\left\{n \in \mathbb{Z}_{\geqslant 0} \mid \mathrm{Kod}(\overline{\mathcal{M}}_{g,n}) = -\infty\right\}$$

and

$$\zeta(g) := \min\left\{n \in \mathbb{Z}_{\geqslant 0} \mid \mathrm{Kod}(\overline{\mathcal{M}}_{g,n}) \geqslant 0\right\}.$$

In the range $4 \leqslant g \leqslant 11$, the state of the art regarding the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ is summarized in the following theorem:

Theorem 1.36. *([Log03], [FP05], [Far09], [FV13]) The Kodaira dimension of* $\overline{\mathcal{M}}_{g,n}$ *encoded in the functions* ζ *and* f *is described in the table below.*

	4	5	6	7	8	9	10	11
$f(g)$	15	13	15	13	12	10	9	10
$\zeta(g)$	16	?	16	14	?	?	10	11
$h(g)$	16	15	16	15	14	13	11	12.

The function h *is defined by the property: for* $n \geqslant h(g)$*, the moduli space* $\overline{\mathcal{M}}_{g,n}$ *is of general type.*

For genus grater than 11, the complete account is summarized in [Far09, Thm 1.10]. One should add to the referred theorem, the results in [Kad17, Thm 1.2], where it is proved that the moduli spaces $\overline{\mathcal{M}}_{16,8}$, $\overline{\mathcal{M}}_{17,8}$ and $\overline{\mathcal{M}}_{12,10}$ are of general type.

An interesting question that, as far a we know, has not been investigated in depth is about the Kodaira dimension of quotients $\overline{\mathcal{M}}_{g,n}/G$ by different subgroups of the symmetric group $G \leqslant \Sigma_n$, that is, the beheavor of the numerical function

$$k(g, n, G) := \mathrm{Kod}\left(\overline{\mathcal{M}}_{g,n}/G\right).$$

For instance, when $n \geqslant g+1$, the full symmetric quotient admits a fibration to the universal Jacobian

$$\begin{array}{rcl} \mathcal{M}_{g,n}/\Sigma_n & \to & \mathscr{J}ac_g^n \\ [C, p_1 + \ldots + p_n] & \mapsto & (C, \mathcal{O}_C(\sum p_i)), \end{array}$$

whose fibers are projective spaces, given by the linear systems $|\sum p_i|$. In particular, for $n \geqslant g+1$, our numerical function $k(g, n, \Sigma_n) = -\infty$. At the other extreme, we have $k(g, n, \{1\}) = \mathrm{Kod}(\overline{\mathcal{M}}_{g,n})$ and, for those pairs (g, n), such that $\overline{\mathcal{M}}_{g,n}$ is of general type, the problem is particularly interesting. This approach might help us understand the transition phenomenon from negative to positive Kodaira dimension in moduli of curves. We will briefly treat some cases at the end of Chapter 3.

1.5.2 Geometry and Kodaira dimension of $\mathcal{F}_g$

For the moduli space of curves $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$, even though different compactifications exists, the one constructed by Deligne and Mumford [DM69], that allows curves to acquire nodes but in within the stable range, is considered central or natural. This is due to mainly two reasons. The fist one has to do with the structure of the boundary and its beautiful stratification and, the second one, has to do with the fact that, for $g \geqslant 4$ and $n \geqslant 0$, the singularities of $\overline{\mathcal{M}}_{g,n}$ are "almost" canonical, in the sense that, if $\widetilde{\mathcal{M}}_{g,n}$ is a desingularization of $\overline{\mathcal{M}}_{g,n}$, then, m-canonical forms on the regular locus

$$\overline{\mathcal{M}}_{g,n}^{\mathrm{reg}} := \overline{\mathcal{M}}_{g,n} \setminus \mathrm{Sing},$$

extend uniquely to $\widetilde{\mathcal{M}}_{g,n}$, that is, there is an isomorphism

$$H^0\left(\overline{\mathcal{M}}_{g,n}^{\mathrm{reg}}, K_{\overline{\mathcal{M}}_{g,n}^{\mathrm{reg}}}^{\otimes m}\right) \cong H^0\left(\widetilde{\mathcal{M}}_{g,n}, K_{\widetilde{\mathcal{M}}_{g,n}}^{\otimes m}\right). \tag{1.4}$$

See [HM82, §2] for $n = 0$ and [Log03, Thm. 2.5] for $n \geqslant 0$.

Remark 1.37. Recall that a normal projective variety X has *canonical singularities* if, for any resolution $\pi : \widetilde{X} \to X$, the difference $K_{\widetilde{X}} - \pi^* K_X$ is $\mathbb{Q}$-effective. In [Log03] is stated that $\overline{\mathcal{M}}_{g,n}$ has only canonical singularities. This is not entirely true, since there are points (for instance, a curve with an elliptic tail with automorphism $\mathbb{Z}/6\mathbb{Z}$) that do not satisfy the Reid-Shepherd-Barron-Tai criterion, i.e., points with an orbifold neighborhood, such that m-canonical forms cannot be extended to a resolution without acquiring poles. In any case, these points do not impose restrictions on m-canonical forms defined on the whole $\overline{\mathcal{M}}_{g,n}^{\mathrm{reg}}$.

This represents a fundamental difference between moduli of curves and moduli of higher dimensional polarized varieties, where there is no "canonical" or "natural" compactification. As mentioned in §1.2.1, for these moduli spaces there are several constructions and each carries its own advantages and disadvantages. More importantly, in many of them we cannot ensure the existence of an isomorphism as (1.4), making the estimate for the Kodaira dimension and other birational invariants very hard.

The choice of compact model has a direct impact on the difficulty to show non-negative Kodaira dimension results, but they have little to do with rationality results. The attempt to show that (as expected) for low genus $\mathcal{F}_g$ is rational, unirational or at least uniruled, in most of the cases does not requires a deep understanding of the boundary or technical singularity control.

1.5.3 Mukai models

The main contributions regarding rationality results were developed by Mukai in [Muk88], [Muk92b], [Muk06] and [Muk12], where it is showed that, for $g \leqslant 10$, and $g = 12, 13, 16, 18$ and 20, there exists a projective variety $V_g \subset \mathbb{P}^{N_g}$ and a vector bundle $\mathscr{E}_g$ over V_g, such that the general K3 surface of genus g is given by the vanishing

$$Z(s) \subset V_g$$

of a general global section $s \in H^0(V_g, \mathscr{E}_g)$, where the polarization is given by the restriction of $\mathcal{O}_{V_g}(1)$ to $Z(s)$. For the explicit list of V_g and $\mathscr{E}_g$, see Appendix §5.1. In particular, the induced map

$$\begin{array}{ccc} \mathbb{P}H^0(V_g, \mathscr{E}_g) & \dashrightarrow & \mathcal{F}_g \\ s & \mapsto & (Z(s), \mathcal{O}(1)) \end{array}$$

rationally dominates $\mathcal{F}_g$. From this follows:

Theorem 1.38. *The moduli space of polarized K3 surfaces $\mathcal{F}_g$ is unirational for*

$$g \leqslant 10 \quad \textit{and} \quad g = 12, 13, 16, 18, 20.$$

The genus eleven case is of spacial interest to us. Mori and Mukai showed in [MM83] the uniruledness of $\mathcal{M}_{11}$, by exhibiting a dominant rational map

$$\mathcal{M}_{11} \dashrightarrow \mathcal{F}_{11} \tag{1.5}$$

with rational fibers. Later, it was proven by Chang and Ran that $\mathcal{M}_{11}$ is unirational, showing via (1.5), the unirationality of $\mathcal{F}_{11}$. We will give a detailed treatment of the case $g = 11$ in Chapter 3. One should add [FV18] to the list of rationality results, where Farkas and Verra showed the universal K3 of genus fourteen $\mathcal{F}_{14,1}$ to be rational. It would be very surprising if, for genus $g = 15$ or $g = 17$, the moduli space $\mathcal{F}_g$ is not unirational. In any case, a proof is missing and there is no reason beside heuristics that guarantees the Kodaira dimension of $\mathcal{F}_g$ does not decreases as g increases.

In analogy with curves, one could ask the same questions about the moduli of polarized K3 surfaces with marked points. This space is not hard to construct assuming the existence of a universal family

$$\mathcal{S} \to \mathcal{F}_g.$$

One can define $\mathcal{F}_{g,n}$ as the n-th fiber product of $\mathcal{S}$ over $\mathcal{F}_g$.

Remark 1.39. Such universal family only exists over the open dense subset $\mathcal{F}_g^\circ \subset \mathcal{F}_g$ parametrizing polarized K3s with trivial automorphism group. Throughout this thesis one can assume $\mathcal{F}_{g,n}$ defined as the fiber product of the universal K3 surface over $\mathcal{F}_g^\circ$ or use the formalism of stacks as a black box and accept that $\mathcal{F}_g$, as a stack, carries a universal surface.

1.5.4 $\mathcal{F}_{g,n}$ and further questions.

Inspired by the analogy with curves, we could ask all sorts of questions about the birational geometry of $\mathcal{F}_{g,n}$. One important feature to observe is that the forgetful map

$$\mathcal{F}_{g,n} \to \mathcal{F}_g$$

has fibers isomorphic to the n-th fiber product of a K3 surface and, by easy addition 1.34

$$\mathrm{Kod}(\mathcal{F}_{g,n}) \leqslant 19.$$

In particular when $n \geqslant 1$, the moduli space $\mathcal{F}_{g,n}$ is never of general type. This is a fundamental difference with $\mathcal{M}_{g,n}$, where, for any $g \geqslant 2$, the moduli is of general type for all, but finitely many n's. On the other hand, from Theorem 1.35, one deduces that for any fixed g, the numerical function

$$k(n) := \mathrm{Kod}(\mathcal{F}_{g,n})$$

cannot decrease, $k(n) \leqslant k(n+1)$. This behavior mimics the equivalent for $\mathcal{M}_{g,n}$, in the sense that, if we mark more points, the moduli space increases in complexity.

One of the fundamental questions related to the geometry of $\mathcal{F}_g$ is the structure of its Picard group. It was first conjectured by Maulik and Pandharipande [MP13, Conj. 3] and recently proven in [BLMM17], that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_g)$ is generated by divisors of the form

$$\mathrm{NL}_{h,d} := \left\{ (S,H) \in \mathcal{F}_g \;\middle|\; \begin{array}{l} \text{there is an embedding of a rank two lattice} \\ \mathbb{Z}H \oplus \mathbb{Z}D \hookrightarrow \mathrm{Pic}(S) \\ \text{with } H \cdot D = d \text{ and } D^2 = 2h-2 \end{array} \right\},$$

subject to the numerical condition $d^2 - 4(g-1)(h-1) > 0$ coming from the Hodge Index Theorem. It would be interesting to give a precise expression of $K_{\mathcal{F}_g}$ in terms of NL-divisors. A. Peterson [Pet15] has partial results in this direction.

1.5.5 Geometry of strata

Despite the fact that, $\mathcal{H}_g^k(\mu)$ has broad connections with several areas in mathematics, and has been studied with an enormous variety of tools, very little is known about its global geometry. For instance, Chen in [Che17a] addresses the question, whether $\mathcal{H}_g^k(\mu)$ contains complete curves. He manages to show that $\mathcal{H}_g^k(\mu)$ does not contain complete curves if some entry of μ satisfies $m_i \leqslant -k$. The question for holomorphic partitions is still open. Unlike $\mathcal{M}_g$, this result raises the possibility for $\mathcal{H}_g^k(\mu)$ to be affine for certain partitions.

A modular interpretation of the boundary $\partial\overline{\mathcal{H}}_g(\mu)$ of the Zariski closure

$$\overline{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}_{g,n}$$

has been the subject of much recent attention. Farkas and Pandharipande [FP15] constructed a proper moduli space called moduli space of *twisted canonical divisors*, that it contains $\mathcal{H}_g(\mu)$ as an open subset. However, the question of which twisted canonical divisors in the boundary lie in $\overline{\mathcal{H}}_g(\mu)$ requires more information than the dual graph of the curve and the relations in the Jacobian of each component. The boundary components of the Zariski closure $\overline{\mathcal{H}}_g(\mu)$ are parametrized, not just by the topological type of the nodal curves and strata in smaller genera, but also by the particular complex structure, manifesting as residue conditions at the nodes provided by Bainbridge, Chen, Gendron, Grushevsky and Möller [BCGGM16a]. The five authors generalized their results for $k \geqslant 1$, cf. [BCGGM16b], providing with a similar description of the boundary of $\overline{\mathcal{H}}^k_g(\mu)$ inside $\overline{\mathcal{M}}_{g,n}$. The question about the geometry of the boundary $\partial\overline{\mathcal{H}}^k_g(\mu)$ is still open. For instance whether it has canonical singularities or, if singular, how to resolve it.

Another interesting open problem is to describe the Picard group of $\mathcal{H}^k_g(\mu)$ and $\overline{\mathcal{H}}^k_g(\mu)$. In [Che17b], Chen provides a set of generators for

$$R^1(\mathcal{H}^k_g(\mu)) := i^*\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{g,n}),$$

where i is the inclusion $\mathcal{H}^k_g(\mu) \subset \mathcal{M}_{g,n}$. It is also known that, divisorial components of the boundary $\partial\overline{\mathcal{H}}^k_g(\mu)$ are parameterized by 2-level graphs, independent of the number of nodes (see [BCGGM16b, §6]) and some relations among them are expected. Still, we know barely nothing about the full Picard groups

$$\mathrm{Pic}_{\mathbb{Q}}\left(\mathcal{H}^k_g(\mu)\right) \quad \text{and} \quad \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{H}}^k_g(\mu)\right).$$

Finally, one should mention the conjectural relation for the class of Farkas and Pandharipande's space of twisted canonical divisors $\widetilde{\mathcal{H}}^k_g(\mu)$ with Pixton's cycle and the Double Ramification cycle in the Chaw ring $A^*(\overline{\mathcal{M}}_{g,n})$. Due to dimension reasons, one has to consider the strictly meromorphic case. In the Appendix of [FP15], Janda, Pandharipande, Pixton, and Zvonkine, define the weighted fundamental class of $\widetilde{\mathcal{H}}^k_g(\mu)$,

$$H_{g,\mu} \in A^g(\overline{\mathcal{M}}_{g,n})$$

as a sum of components of $\widetilde{\mathcal{H}}^k_g(\mu)$ with certain multiplicities coming from Gromov-Witten theory. They conjectured that this class equals

$$2^{-g}P^g_{g,\mu} \in R^g(\overline{\mathcal{M}}_{g,n}),$$

famous Pixton's cycle also deeply related with the *double ramification cycle*, cf. [CJ16].

Many other questions and open problems regarding the spaces $\mathcal{H}_g^k(\mu)$ are being currently studied. Our selection says more about the author's background and taste, than the importance of the problems itself.

1.6 Outline of results

The rest of the thesis is organized as follows. In Chapter 2, we develop the deformation theory of nodal curves with marked points embedded in K3 surfaces. We study the moduli map

$$c_{g,\delta,l} : \mathcal{V}_{g,\delta,l} \to \mathcal{M}_{g-\delta,2\delta+l}/\mathbb{Z}_2^{\oplus\delta},$$

from the *universal Severi variety* parameterizing tuples

$$(S, H, X, x_1, \dots, x_\delta, y_1, \dots, y_l),$$

where $(S, H) \in \mathcal{F}_g$ is a polarazied K3 of genus g,

$$x_1, \dots, x_\delta, y_1, \dots, y_l \in S$$

are points on the surface and X is a δ-nodal irreducible curve on the linear system $|H|$, with nodes at the x_i's and passing through the points y_i's. The target of the map $c_{g,\delta,l}$, consist of a quotient, where the i-th factor of $\mathbb{Z}_2$ acts by permuting the $(2i-1)$-th and $2i$-th marked points. In other words, the moduli of δ-nodal curves in $\mathcal{M}_{g,l}$. We denote the $\mathbb{Z}_2^{\oplus\delta}$-quotient by $\mathcal{M}_{g-\delta,[2\delta]+l}$. The aim of Chapter 2 is to establish a range where this map is dominant. We will see that there are moduli maps as in the diagram

$$\begin{tikzcd} & \mathcal{V}_{g,\delta,l} \arrow[dl, "\pi"'] \arrow[dr, "c_{g,\delta,l}"] & \\ \mathcal{F}_{g,\delta+l} & & \mathcal{M}_{g-\delta,[2\delta]+l}, \end{tikzcd}$$

where π is the map that forgets the nodal curve X and $c_{g,\delta,l}$ is the map induced by the normalization. The main results proved in Chapter 2 are the following theorems:

Theorem 1.40. *In the range* $3 \leqslant g \leqslant 11$, $0 \leqslant \delta \leqslant g-2$ *and* $g \neq 10$, *the moduli map*

$$c_{g,\delta,l} : \mathcal{V}_{g,\delta,l} \to \mathcal{M}_{g-\delta,[2\delta]+l}$$

is dominant and the dimension of its general fiber is equal to $22-2g$.

Regarding the map on the left:

Theorem 1.41. *For g and δ in the same range as above, the map π is dominant when $3\delta + l \leqslant g$ and birational when $3\delta + l = g$.*

In Chapter 3 we concentrate on the birational geometry of $\mathcal{F}_{11,n}$. We prove the following theorem:

Theorem 1.42. *The moduli space $\mathcal{F}_{11,n}$*

- *is unirational for $n \leqslant 6$,*
- *is uniruled for $n \leqslant 7$ and*
- *it has non-negative Kodaira dimension for $n \geqslant 9$.*

We will show that the nomalization map $c_{g,\delta,l}$ is birational when $g = 11$. As a consequence we will construct different birational models for low genus curves with marked points and K3 surfaces in genus eleven. More explicitly, we prove:

Theorem 1.43. *For $\delta \leqslant 9$, there is a rational map*

$$\mathcal{M}_{11-\delta,2\delta+l}/\mathbb{Z}_2^{\oplus\delta} \dashrightarrow \mathcal{F}_{11,\delta+l},$$

dominant for $3\delta + l \leqslant 11$ and birational for $3\delta + l = 11$.

In the last section of Chapter 3, we discuss further applications. Specifically, we discuss how this picture can be used to attack some of the missing cases in the Kodaira classification of $\mathcal{M}_{g,n}$ for small g, such as $\mathcal{M}_{9,11}$, $\mathcal{M}_{9,12}$ and $\mathcal{M}_{10,10}$.

In the last chapter we investigate the birational geometry of the strata $\mathcal{H}_g^k(\mu)$ for $k = 1$, $k = 2$ and μ holomorphic partition of $k \cdot (2g-2)$. For $k = 1$, we show the following theorem regarding its birational geometry.

Theorem 1.44. *Let $\overline{\mathcal{H}}_g(\mu)$ be an irreducible stratum with length of partition $l(\mu)$. The birational geometry of $\overline{\mathcal{H}}_g(\mu)$ is summarized in the following table:*

Table 1.1

	Unirational	Uniruled
$3 \leqslant g \leqslant 6$	$l(\mu) \leqslant g-1$	No restriction on μ
$g = 7, 8$	?	No restriction on μ
$g = 9$	?	$l(\mu) \geqslant 7$
$g = 10$	?	$11 \leqslant l(\mu) < 18$
$g = 11$	?	$l(\mu) \geqslant 10$

For quadratic differentials (i.e., $k = 2$) we prove the following:

Theorem 1.45. *Let $\overline{\mathcal{H}}_g^2(\mu)$ be an irreducible stratum of holomorphic quadratic differentials. We assume μ to be a primitive partition of length $l(\mu)$. Then, for genus $3 \leqslant g \leqslant 6$ and $l(\mu) \geqslant g$, the moduli space $\overline{\mathcal{H}}_g^2(\mu)$ is uniruled.*

We add some results concerning the non-irreducible stratum.

In the last section of Chapter 4, we study the forgetful maps

$$\mathcal{H}_g^k(\mu) \to \mathcal{M}_{g,n},$$

when the length of the partition μ ensures generic finiteness. The degrees of these maps are already known to experts, but the lack of a reference inspired us to give a full computation of these degrees. Finally, we give an expression for the canonical class of the open stratum $\mathcal{H}_g^k(\mu)$ in terms of pull back of divisor classes in $\mathcal{M}_{g,l(\mu)}$. We do this by studying the ramification of these maps together with some relations in the Picard group of $\mathcal{H}_g^k(\mu)$. We prove the following theorem:

Theorem 1.46. *Let $\mu = (m_1, \dots, m_{l(\mu)})$ be a partition of length $l(\mu) \geqslant g$. We assume the partition μ to be meromorphic if $k = 1$ and primitive if $k \geqslant 2$. Then, the canonical class of the open strata is given by*

$$K_{\mathcal{H}_g^k(\mu)} = c_\mu \lambda,$$

where

$$c_\mu = 12 + \frac{12k^2 \left(\sum \frac{1}{k+m_i} \right)}{k \cdot (4g-4) - \sum \frac{k^2+m_i^2}{k+m_i}} \in \mathbb{Q}$$

if $m_i \neq -k$ for all $i = 1, \dots, l(\mu)$, and

$$c_\mu = 6$$

if some $m_i = -k$.

We expect that these formulas can be extended to the boundary with the hope that some positive Kodaira dimension results can be achieved.

This thesis is a more detailed exposition of the results contained in two submitted preprints; [Bar17b] and [Bar17a]. The first paper contains most of the results and exposition in Chapter 4 and a weaker version of the results in Chapter 2, that we used in [Bar17b] to treat uniruledness of the strata in genus 10. The second paper contains a full proof of the results in Chapter 2 and most of the results in Chapter 3. The computations regarding the canonical class of the strata constitutes unpublished material that I expect to extend and use in the future. Also the comments on the Kodaira dimension of $\mathcal{M}_{9,11}$, $\mathcal{M}_{9,12}$ and $\mathcal{M}_{10,10}$ are unpublished. We are planing to investigate these further, hoping to get stronger results.

CHAPTER 2

Nodal curves on K3 surfaces

2.1 Introduction

The aim of this chapter is to study the deformation theory of nodal curves on K3 surfaces. Curves lying on K3 surfaces are important for many reasons and the smooth case has been studied by many with very fruitful outcomes. One of the main interests in smooth curves lying on K3 surfaces is that, in many contexts, they behave as if they where general curves. For instance, the Brill-Noether theory of a smooth hyperplane section of a K3 surface with Picard number one is the same as the theory for a general curve, [Laz86]. Another example is the use of curves in K3 surfaces to understand the syzygies of a general curve which ultimately lead to the proof of many instances of Green's conjecture; [Voi02a], [Voi05], [FK16] and [FK17].

Let $\mathcal{F}_{g,n}$ be the moduli space of polarized K3 surfaces constructed in section §1.3. When there are no marked points we omit "n" in the notation. There is a natural projective bundle $\mathcal{P}_g$ over $\mathcal{F}_g$, whose fiber over a point (S,H) is the projective space of linear sections $|H|$. By adjunction formula, the genus of a smooth curve in $|H|$ is g. On a similar fashion, one can define $\mathcal{P}_{g,n}$ over $\mathcal{F}_{g,n}$ with fiber over $(S,H,x_1,\dots,x_n)$ being the space of hyperplane sections $C\in|H|$ passing though the points $x_1,\dots,x_n\in S$, that is, the projective (possibly empty) space

$$\mathbb{P}\left(H^0\left(S,H\otimes\mathcal{I}_{x_1+\ldots+x_n}\right)^\vee\right),$$

where $\mathcal{I}_{x_1+\ldots+x_n}$ is the ideal sheaf of the zero dimensional scheme defined by the points. We define $\mathcal{U}_{g,n}\subset\mathcal{P}_{g,n}$ to be

$$\mathcal{U}_{g,n}:=\left\{(S,H,C,x_1,\dots,x_n)\;\middle|\;\begin{array}{l}(S,H,x_1,\dots,x_n)\in\mathcal{F}_{g,n}\text{ and}\\ C\in\mathbb{P}\left(H^0\left(S,H\otimes\mathcal{I}_{x_1+\ldots+x_n}\right)^\vee\right)\text{ smooth.}\end{array}\right\}$$

There are two natural moduli maps

$$\begin{tikzcd} & \mathcal{U}_{g,n} \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\ \mathcal{F}_{g,n} & & \mathcal{M}_{g,n}. \end{tikzcd}$$

As we saw in the introduction, the dimension of $\mathcal{F}_{g,n}$ is equal to $19 + 2n$ and, when $n \leqslant g$ and the points $x_1, \dots, x_n$ are general on S, the dimension of the fiber of π is $g - n$. Thus, there is an open subset of $\mathcal{P}_{g,n}$ of dimension $19 + g + n$. Only when $g = 11$ and $n \leqslant 11$, the dimension the source and target of ϕ are equal. Mori and Mukai proved in [Muk96], [MM83] that when $g = 11$, the map ϕ is not only finite but also birational, meaning, a general genus eleven curve $[C] \in \mathcal{M}_{11}$ lies in a unique K3 surfaces as hyperplane section. One of the main goals of this thesis is to study the domain of definition of ϕ. For general (S, H), the locus of δ-nodal curves is of codimension δ in the linear system $|H|$. On the other hand δ-nodal curves define a codimension δ locus in $\overline{\mathcal{M}}_g$. We will see that the birational isomorphism extends to the nodal locus, giving rise to a collection of birational isomorphisms between *universal Severi varieties* and the moduli of δ-nodal curves of fixed genus. These results establish a deep connection between K3 surfaces of genus 11 with marked points and moduli of curves of genus less than or equal to eleven, also with marked points.

Definition 2.1. Let $(S, H, x_1, \dots, x_\delta) \in \mathcal{F}_{g,\delta}$ and $\delta \leqslant g$. We define

$$V_\delta(S, H, x_1, \dots, x_\delta) \subset |H|$$

to be the *Severi variety* of irreducible δ-nodal curves in $|H|$ with nodes at $x_1, \dots, x_\delta$. We define the *universal Severi variety* $\mathcal{V}_{g,\delta}$ to be the algebraic stack parametrizing tuples $(S, H, X, x_1, \dots, x_\delta)$, where $(S, H, x_1, \dots, x_\delta) \in \mathcal{F}_{g,\delta}$ and $X \in V_\delta(S, H, x_1, \dots, x_\delta)$.

The stack $\mathcal{V}_{g,\delta}$ is smooth and every irreducible component has dimension $19 + (g - \delta)$. It was conjectured by Ciliberto and Dedieu [CD12, Thm. 2.1] that the universal Severi variety with unordered nodes, i.e., the symmetric quotient

$$\mathcal{V}_{g,\delta}/\Sigma_\delta,$$

is irreducible. They managed to prove it in the range $3 \leqslant g \leqslant 11$, $g \neq 10$ and $0 \leqslant \delta \leqslant g$.

We will denote by

$$\mathcal{M}_{g,[2m]+n} := \mathcal{M}_{g,2m+n}/\mathbb{Z}_2^{\oplus m},$$

the $\mathbb{Z}_2^{\oplus m}$-quotient of the moduli space of smooth curves of genus g with $2m+n$ marked points, where the i-th copy of $\mathbb{Z}_2$ acts permuting the $2i-1$ with the $2i$-th marked point. A standard element is denoted by

$$[C, p_1 + q_1, \dots, p_m + q_m, y_1, \dots, y_n] \in \mathcal{M}_{g,[2m]+n}.$$

There are well defined moduli maps

$$\begin{tikzcd} \mathcal{V}_{g,\delta} \arrow[r, "c_{g,\delta}"] \arrow[dr, "c"'] & \mathcal{M}_{g-\delta,[2\delta]} \arrow[d] \\ & \mathcal{M}_{g-\delta}, \end{tikzcd}$$

where $c_{g,\delta}$ sends a pair $(S, X, x_1, \dots, x_\delta)$ to the isomorphism class of the normalization of X together with the preimages of the nodes and the vertical arrow is the usual forgetful map remembering only the curve. For $\delta = 0$, the map c is Mukai's map, dominant for $g \leqslant 9$ and $g = 11$, not dominant for $g = 10$ and generically finite over the image for $g \geqslant 11$ and $g \neq 12$. See [Muk88], [Muk96], [Muk92a] and [MM83]. Moreover, for $g = 11$ and $g \geqslant 13$, Mukai's map is birational over its image, cf. [CLM93, Thm. 4.5].

In [Bea04, §5], Beauville studied the differential of $c_{g,0}$ and gave a deformation theoretic proof of Mukai's results. He sowed that for a general element $(S, H, C) \in \mathcal{V}_{g,0}$, the cokernel of the differential

$$dc_{g,0} : T_{(S,H,C)}\mathcal{V}_{g,0} \to T_{[C]}\mathcal{M}_g$$

is isomorphic to

$$H^0(S, \Omega^1_S(H))$$

and computed these cohomology groups.

Proposition 2.2 (cf. §5.2 in [Bea04])**.** *For general* $(S, H) \in \mathcal{F}_g$ *with* $g \geqslant 2$,

- $h^0(S, \Omega^1_S(H)) = 0$ *for* $g \leqslant 11, g \neq 10$.
- $h^0(S, \Omega^1_S(H)) = 1$ *for* $g = 10$.

By a similar approach, but to the nodal case, Flamini, Knutsen, Pacienza and Sernesi proved in [FKGS08] that for the range $3 \leqslant g \leqslant 11$ and $0 \leqslant \delta \leqslant g-2$, the moduli map

$$c : \mathcal{V}_{g,\delta} \to \mathcal{M}_{g-\delta}$$

is dominant with general fiber of dimension $22 - 2(g - \delta)$. We will show that by keeping track of the nodes, the geometry of the moduli map $c_{g,\delta}$ is much richer, giving rise no only to finite but birational maps.

Definition 2.3. We call $\mathcal{V}_{g,\delta,l}$ the moduli space of irreducible δ-nodal curves with l marked points on a polarized K3 surface,

$$\mathcal{V}_{g,\delta,l} := \left\{ (S,H,X,x_1,\ldots,x_\delta,y_1,\ldots,y_l) \;\middle|\; \begin{array}{l} (S,H) \in \mathcal{F}_g, \\ X \in |H| \text{ irreducible}, \\ \delta\text{-nodal at } x_1,\ldots,x_\delta \\ \text{and } y_1,\ldots,y_l \in X. \end{array} \right\}.$$

The moduli $\mathcal{V}_{g,\delta,l}$ is defined as the fiber product

$$\mathcal{V}_{g,\delta,l} := \mathcal{V}_{g,\delta} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,l},$$

where $\overline{\mathcal{M}}_{g,l} \to \overline{\mathcal{M}}_g$ is the natural forgetful map and

$$\mathcal{V}_{g,\delta} \to \overline{\mathcal{M}}_g$$

is the map that remembers the nodal curve. There are two natural moduli maps to consider

$$\begin{tikzcd} & \mathcal{V}_{g,\delta,l} \arrow[dl, "\pi"'] \arrow[dr, "c_{g,\delta,l}"] & \\ \mathcal{F}_{g,\delta+l} & & \mathcal{M}_{g-\delta,[2\delta]+l}, \end{tikzcd}$$

where π is the map that forgets the nodal curve X and $c_{g,\delta,l}$ is the map induced by the normalization. The aim of this chapter is to show the following theorems.

Theorem 2.4. *In the range* $3 \leqslant g \leqslant 11$, $0 \leqslant \delta \leqslant g-2$ *and* $g \neq 10$, *the moduli map*

$$c_{g,\delta,l} : \mathcal{V}_{g,\delta,l} \to \mathcal{M}_{g-\delta,[2\delta]+l}$$

is dominant with general fiber dimension equals to $22-2g$.

And regarding the map on the left:

Theorem 2.5. *For* g *and* δ *in the same range as above, the map* π *is dominant when* $3\delta + l \leqslant g$ *and birational when* $3\delta + l = g$.

Remark 2.6. Consider the following diagram:

$$\begin{tikzcd} \mathcal{V}_{10,1} \arrow[r, "c_{10,1}"] \arrow[dr, "c"'] & \mathcal{M}_{10-\delta,[2]} \arrow[d] \\ & \mathcal{M}_9 \end{tikzcd}$$

The map c is dominant, but the map $c_{10,1}$ is not. For a general curve C of genus nine, the condition on

$$p + q \in \mathrm{Sym}^2 C$$

so that the nodal curve $C/p \sim q$ admits a K3 extension of genus ten is divisorial. It was asked in [FKGS08] what is the class of such locus. One can phrase the question in a slightly more general way. The situation is the same when we allow more nodes. Indeed, for $1 \leqslant \delta \leqslant 8$, the image of the map $c_{10,\delta,l} : \mathcal{V}_{10,\delta,l} \to \mathcal{M}_{10-\delta,[2\delta]+l}$ is always of codimension one and when we composed with the forgetful map $\mathcal{M}_{10-\delta,[2\delta]+l} \to \mathcal{M}_{10-\delta,l}$, the map is dominant. Let C be a general genus $10-\delta$ curve. What is the class in $\left(\mathrm{Sym}^2 C\right)^{\oplus\delta}$ of the locus of points $(p_1+q_1, \dots, p_\delta+q_\delta)$ such that the nodal curve

$$C/\{p_i \sim q_i\}_{i=1,\dots,\delta}$$

admits a K3 extension of genus 10? Or more generally, one can ask for the class of the image of $c_{10,\delta,l}$ in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{10-\delta,[2\delta]+l})$. This class is easy to compute and can be seen as the intersection of the class of the K3 locus in $\overline{\mathcal{M}}_{10,l}$ computed by Farkas and Poppa in [FP05, Thm 1.6], with the boundary strata of δ-nodal irreducible curves. See Corollary 2.14.

Of particular interest to us is the genus eleven case. In the next chapter we will show the map $c_{11,\delta,l}$ is not only to be dominant, but birational for $0 \leqslant \delta \leqslant 9$.

2.2 Deformation theory

Deformation theory is a vast and very rich subject that, even though deserves to be study as a subject by its own, it is an essential tool when it comes to understanding the local structure of moduli spaces. Using the functorial approach, one can see that, if we identify X with its functor of points, an element of $X(\mathrm{Spec}\, \mathbb{C}[\varepsilon]/(\varepsilon^2))$ correspond to the choice of a closed point $x \in X$ and a vector on the tangent space of X at x. This motivates the definition of *first order deformations*. Recall that we want to study moduli of curves in K3 surfaces.

Let $j : X \hookrightarrow S$ be a closed embedding of a curve (possibly nodal) on a smooth surface and

$$\mathbb{C}[\varepsilon] := \mathbb{C}[\varepsilon]/(\varepsilon^2)$$

the ring of *dual numbers*.

Definition 2.7. A *first order deformation* of j is a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"] & \mathscr{X} \arrow[d, "J"] \\ S \arrow[r] \arrow[d] & \mathscr{S} \arrow[d, "\phi"] \\ \mathrm{Spec}\,(\mathbb{C}) \arrow[r, hook, "s"] & \mathrm{Spec}\,(\mathbb{C}[\varepsilon]), \end{tikzcd}$$

where ϕ and $\phi \circ J$ are flat and the horizontal arrows induce isomorphisms with the pull back of the right column by s. A first order deformation is *locally trivial* if there exist an affine open cover $\{U_i\}_{i \in I}$ of S, such that $V_i = X_i \cap X$ is affine and the diagram above restricts to a trivial deformation, that is,

$$\begin{tikzcd} V_i \arrow[r] \arrow[d, "j"] & V_i \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]) \arrow[d, "j \times Id"] \\ U_i \arrow[r] \arrow[d] & U_i \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]) \arrow[d, "\phi"] \\ \mathrm{Spec}\,(\mathbb{C}) \arrow[r, hook, "s"] & \mathrm{Spec}\,(\mathbb{C}[\varepsilon]). \end{tikzcd}$$

Given another first order deformation of j

$$\begin{tikzcd} \mathscr{X}' \arrow[rr, "J'"] \arrow[dr] & & \mathscr{S}' \arrow[dl] \\ & \mathrm{Spec}\,(\mathbb{C}[\varepsilon]), & \end{tikzcd}$$

an *isomorphism of first order deformations* is a pair of $\mathbb{C}[\varepsilon]$-isomorphisms

$$\mathscr{X} \xrightarrow{\sim} \mathscr{X}', \mathscr{S} \xrightarrow{\sim} \mathscr{S}'$$

that restrict to the identity on the central fiber and makes the diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\sim"] \arrow[d, "J"] & \mathscr{X}' \arrow[d, "J'"] \\ \mathscr{S} \arrow[r, "\sim"] & \mathscr{S}' \end{tikzcd}$$

commutative.

The set if isomorphism classes

$$\mathrm{Def}(j) := \{\text{Locally trivial first order deformations of } j\} \big/ \text{isom}$$

has a natural $\mathbb{C}$-vector space structure.[1] If X is smooth, every first order deformation of it is locally trivial. This cease to be true when we allow nodes and locally trivial deformations are those that keep the nodes. Since we will consider a moduli space of nodal curves on K3 surfaces, the local triviality assumption is essential. As usual in deformation theory, out of a locally trivial deformation one can construct, in a canonical way, a Čech cocycle that induces an identification between $\mathrm{Def}(j)$ and a cohomology group of certain sheaf. We highlight the main steps of the construction, delegating details to the better written sources in the subject, e.g. [Ser06].

Let $j : X \to S$ be a closed embedding as above $\{U_i\}$ an affine open cover of S with $V_i = X \cap U_i$ the induced affine open cover of X. Every locally trivial first order deformation $J \in \mathrm{Def}(j)$ is obtained by gluing the trivial deformations

$$\begin{tikzcd} V_i \arrow[r, hook] \arrow[d, hook, "j"] & V_i \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]) \arrow[d, hook] \\ U_i \arrow[r, hook] & U_i \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]) \end{tikzcd}$$

along the intersections $U_{ij} = U_i \cap U_j$, $V_{ij} = V_i \cap V_j$. Every such gives us pairs of local automorphisms

$$\theta_{ij} \in \mathrm{Aut}\left(V_{ij} \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon])\right), \Theta_{ij} \in \mathrm{Aut}\left(U_{ij} \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon])\right),$$

making the following diagram commutative:

$$\begin{tikzcd} V_{ij} \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]) \arrow[r, "\theta_{ij}"] \arrow[d, hook] & V_{ij} \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]) \arrow[d, hook] \\ V_{ij} \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]) \arrow[r, "\Theta_{ij}"] & V_{ij} \times \mathrm{Spec}\,(\mathbb{C}[\varepsilon]). \end{tikzcd}$$

One can show that Θ_{ij} and θ_{ij} correspond to sections $D_{ij} \in \Gamma(U_{ij}, T_S)$ and $d_{ij} \in \Gamma(V_{ij}, T_X)$ respectively such that D_{ij} restricts to d_{ij}. Moreover D_{ij} and d_{ij} satisfy a cocycle condition inducing a Čech cocycle in $H^1(S, T_S)$ and $H^1(X, T_X)$ respectively. Here T_X stands for the dual sheaf of Ω^1_X.

Definition 2.8. We define the *sheaf of germs of tangent vectors to* S *that are tangent to* X, denoted $T_S\langle X\rangle$, to be the inverse image of $T_X \subset T_S\,|_X$ under the natural restriction map $T_S \to T_S\,|_X$.

It follows from the definition that $D_{ij} \in \Gamma(U_{ij}, T_S\langle X\rangle)$. Locally trivial deformations of the pair (S, X) are governed by the sheaf $T_S\langle X\rangle$, more specific

$$\mathrm{Def}(X \hookrightarrow S) \cong H^1(S, T_S\langle X\rangle).$$

[1] We are skipping mayor parts of the general theory, for detailed and more general exposition we recommend [Ser06].

Remark 2.9. The sheaf $T_S\langle X\rangle$ is a locally free subsheaf of T_S whose restriction to X is the tangent sheaf T_X. When $p \in X$ is a node with local coordinates $\{xy = 0\}$, the localization is

$$T_S\langle X\rangle_p = \mathcal{O}_{S,p}x\frac{\partial}{\partial x} \oplus \mathcal{O}_{S,p}y\frac{\partial}{\partial y}.$$

An alternative definition is

$$T_S\langle X\rangle = \left(\Omega^1_S(\log X)\right)^{\vee},$$

where $\Omega^1_S(\log X)$ is the sheaf of meromorphic 1-forms with at worst logarithmic poles along X.

The sheaf $T_S\langle X\rangle$ sits in two standard exact sequences

$$0 \to T_S(-X) \to T_S\langle X\rangle \to T_X \to 0$$

and

$$0 \to T_S\langle X\rangle \to T_S \to N'_{X/S} \to 0, \tag{2.1}$$

where $N'_{X/S}$ is the *equisingular normal sheaf* of X in S. This sheaf governs the deformation theory when S is fixed. Moreover,

$$T_X V_\delta(S, H) = H^0(X, N'_{X/S}).$$

The first order locally trivial deformations of the pair (S, X) are parametrize by $H^1(S, T_S\langle X\rangle)$. Obstructions are parametrize by H^2 and local automorphisms by H^0. The theory is unobstructed and the coarse moduli space $\mathcal{V}_{g,\delta}$ is smooth as stack [FKGS08, Prop 4.8]. Moreover, for any (S, X)

$$h^2(T_S\langle X\rangle) = h^0(T_S\langle X\rangle) = 0 \tag{2.2}$$

and

$$T_{(S,X)}\mathcal{V}_{g,\delta} = H^1(S, T_S\langle X\rangle).$$

Much more can be said. Given such a pair (S, X) there is a unique embedded resolution of X given by the following diagram

$$\begin{tikzcd} C \cap E \arrow[r, hook] \arrow[d] & C \arrow[r, hook] \arrow[d, "f"] & \tilde{S} \arrow[d, "\varepsilon"] \\ \mathrm{Sing}(X) \arrow[r, hook] & X \arrow[r, hook] & S, \end{tikzcd}$$

where $\tilde{S}$ is the blow-up of S along the nodes, $E = E_1 + \ldots + E_\delta$ is the exceptional divisor, C is a smooth genus $g-\delta$ curve and $f : C \to X \subset S$ is the normalization map. Let us take a look at the tangent exact sequence for the normalization map

$$0 \to T_C \to f^*T_S \to N_f \to 0.$$

Here N_f is the normal sheaf of the map $f : C \to S$. With this notation [FKGS08, Lemma 4.16]

$$f_*(N_f) = N'_{X/S} \text{ and } H^i(N'_{X/S}) \cong H^i(N_f) \text{ for } i = 0, 1. \tag{2.3}$$

This is not surprising since the group $H^0(N_f)$ can be identified with the tangent space at [f] of the space of maps $f : C \to S$ from genus $g - \delta$ smooth curves to a fixed target with $f_*C = H$. There is a one to one correspondence between δ-nodal curves on S in the linear system $|H|$ and maps f. The correspondence is given by normalization

$$\begin{array}{rcl} V_\delta(S, H) & \to & M_{g-\delta}(S, H) \\ X \subset S & \mapsto & f : C \to S. \end{array}$$

To recover the differential of the map $c_{g,\delta}$ we go to $\tilde{S}$. Consider the following diagram as in [FKGS08],

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ & \varepsilon^* T_S(-C) \arrow[r, "\cong"] \arrow[d] & \varepsilon^* T_S(-C) \arrow[d] & & \\ 0 \arrow[r] & \mathscr{F}_C \arrow[r] \arrow[d, "\tau"] & \varepsilon^* T_S \arrow[r, "\lambda"] \arrow[d] & N_f \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T_C \arrow[r] \arrow[d] & f^* T_S \arrow[r] \arrow[d] & N_f \arrow[r] & 0, \\ & 0 & 0 & & \end{tikzcd} \tag{2.4}$$

where $\mathscr{F}_C$ is the kernel of the composition $\lambda : \varepsilon^* T_S \to f^* T_S \to N_f$. Turns out [FKGS08, Prop. 4.22] that

$$\varepsilon_* \mathscr{F}_C \cong T_S\langle X \rangle \quad \text{and} \quad H^i(S, T_S\langle X \rangle) \cong H^i(\tilde{S}, \mathscr{F}_C) \text{ for } i = 0, 1, 2. \tag{2.5}$$

Moreover, $H^1(\tau)$ is the differential of the map $c_{g,\delta}$. The four authors proved that in the desired range, for a general choice $H^1(\tau)$ is surjective, see [FKGS08, Thm. 5.1].

Proposition 2.10. *With the same notation as above* $\mathscr{F}_C \mid_E \cong \mathcal{O}_E(-1)^{\oplus 2}$.

Proof. Let $j : E_i \hookrightarrow \tilde{S}$ be the close embedding of one of the components of the exceptional divisor. Notice that $\varepsilon^* T_S \mid_{E_i} \cong \mathcal{O}_{E_i}^{\oplus 2}$ and $N_f \cong \omega_C$. From the second row in (2.4) we have

$$0 \to L^1 j^* \omega_C \to \mathscr{F}_C \mid_{E_i} \to \mathcal{O}_{E_i}^{\oplus 2} \to \mathcal{O}_{E_i \cap C} \to 0$$

and $C \cap E = p_i + q_i$. Then

$$\deg(\mathscr{F}_C\mid_{E_i}) = \deg L^1 j^* \omega_C - 2.$$

On the other hand by adjuction $\omega_C \cong \mathcal{O}_C(E+C)$ and pulling back by j the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(E) \to \mathcal{O}_{\tilde{S}}(E+C) \to \omega_C \to 0$$

we get

$$0 \to L^1 j^* \omega_C \to \mathcal{O}_{E_i}(E_i) \to \mathcal{O}_{E_i}(E_i + C) \to \mathcal{O}_{p_i+q_i} \to 0.$$

Counting degrees we have $\deg \mathcal{O}_{E_i}(E_i) = -1$ and $\deg \mathcal{O}_{E_i}(E_i + C) = 1$. Thus, $\deg L^1 j^* \omega_C = 0$ and $\deg \mathscr{F}_C\mid_{E_i} = -2$. The sheaf $\mathscr{F}_C\mid_{E_i}$ is free on E_i of rank two and degree -2. By Riemann-Roch,

$$h^0\left(E_i, (\mathscr{F}_C\mid_{E_i}\right) = h^1\left(E_i, \mathscr{F}_C\mid_{E_i}\right).$$

Its enough to prove $h^0 = h^1 = 0$. By pushing forward the second row in (2.4), we get the exact sequence (2.1) and by (2.3) together with (2.5), we obtain

$$R^1 \varepsilon_* \mathscr{F}_C = 0.$$

The exceptional locus of ε is one dimensional, therefore $R^2 \varepsilon_* \mathscr{F}_C(-E_i) = 0$. Thus, from

$$0 \to \mathscr{F}_C(-E_i) \to \mathscr{F}_C \to \mathscr{F}_C\mid_{E_i} \to 0$$

we get $R^1 \varepsilon_* \mathscr{F}_C\mid_E = 0$. In particular $h^1(E_i, \mathscr{F}_C\mid_{E_i}) = h^0(E_i, \mathscr{F}_C\mid_{E_i}) = 0$. □

We have the following corollary:

Corollary 2.11. *The inclusion $\mathscr{F}_C(-E) \subset \mathscr{F}_C$ induces isomorphisms*

$$H^i(\tilde{S}, \mathscr{F}_C(-E)) \cong H^i(\tilde{S}, \mathscr{F}_C) \text{ for } i = 0, 1, 2.$$

In particular $H^1(\mathscr{F}_C(-E))$ parametrizes locally trivial first order deformations of the pair (S, X).

Proof. If we pass to cohomology in the short exact sequence

$$0 \to \mathscr{F}_C(-E) \to \mathscr{F}_C \to \mathscr{F}_C\mid_E \to 0,$$

since $H^0(\mathscr{F}_C\mid_E) \cong H^1(\mathscr{F}_C\mid_E) = 0$, we obtain our result. □

Consider the exact sequence obtained by tensoring the first column (2.4) with $\mathcal{O}_{\tilde{S}}(-E)$,

$$0 \to \varepsilon^* T_S(-C-E) \to \mathscr{F}_C(-E) \xrightarrow{\sigma} T_C\left(-\sum_{i=1}^{\delta} p_i + q_i\right) \to 0.$$

Lemma 2.12. *The following isomorphism of sheaves holds:*

$$f_*T_C\left(-\sum_{i=1}^{\delta} p_i + q_i\right) \cong T_X.$$

Proof. The local equation around each node of X is given by

$$\{xy = 0\} \subset U \subset \mathbb{A}^2,$$

and Ω_U is generated by dx, dy with the relation $xdy + ydx = 0$. The sheaf $T_X = \mathrm{Hom}(\Omega_X, \mathcal{O})$ restricted to U is generated by the maps $(dx, dy) \mapsto (x, 0)$ and $(dx, dy) \mapsto (0, y)$. On the brach defined by x, the first generator is x times the generator $dx \mapsto 1$ of the tangent bundle at that branch and the same for y. This means that locally around the node the normalization map give us an isomorphism $xT_{U_x} \oplus yT_{U_y} \cong T_U$. These local isomorphisms coincide away from the nodes forming a global isomorphism. □

Proposition 2.13. *The map*

$$H^1(\tau) : H^1(\tilde{S}, \mathscr{F}_C(-E)) \to H^1(C, T_C(-p_1 - q_1 - \ldots - p_\delta - q_\delta))$$

is the differential of $c : \mathcal{V}_{g,\delta} \to \mathcal{M}_{g-\delta,[2\delta]}$.

Proof. The blow-up map $\varepsilon : \tilde{S} \to S$ restricted to C is finite so the higher derived images of ε_* are always trivial and by Lemma 2.12, $\varepsilon_* T_C(-\sum p_i + q_i)$ is the tangent sheaf of the nodal curve X. Again if we apply ε_* to the second row in diagram (2.4), from (2.3) and (2.5) we get (see also [FKGS08, Prop. 4.22])

$$R^i\varepsilon_*\mathscr{F}_C \cong R^i\varepsilon_*\varepsilon^*T_S = 0 \text{ for } i > 0.$$

From the exact sequence

$$0 \to \mathscr{F}_C(-E) \to \mathscr{F}_C \to \mathcal{O}_E(-1)^{\oplus 2} \to 0,$$

we can conclude that $R^i\varepsilon_*\mathscr{F}_C(-E) = 0$ for $i > 0$. By Proposition 2.10 and (2.5) we have isomorphisms

$$\varepsilon_*\mathscr{F}_C(-E) \xrightarrow{\sim} \varepsilon_*\mathscr{F}_C \cong T_S\langle X\rangle.$$

Therefore, the natural isomorphisms coming from the Leray spectral sequence sit in the diagram

$$\begin{array}{ccc} H^1(\tilde{S}, \mathscr{F}_C(-E)) & \xrightarrow{H^1(\sigma)} & H^1(C, T_C(-\sum p_i + q_i)) \\ \downarrow\cong & & \downarrow\cong \\ H^1(S, \varepsilon_*\mathscr{F}_C(-E)) & \xrightarrow{H^1(\varepsilon_*\sigma)} & H^1(T_X), \end{array} \tag{2.6}$$

where the map at the bottom factors through a natural isomorphism $H^1(\varepsilon_*\mathscr{F}_C(-E)) \cong H^1(T_S\langle X\rangle)$ and H^1 of the restriction map $T_S\langle X\rangle \to T_X$ sending locally trivial first order deformations of the pair (S, X) to nodal deformations of X. □

As stated in [FKGS08], if $V_{g,\delta} \to \mathcal{V}_{g,\delta}$ is an étale atlas and $\mathscr{X} \hookrightarrow \mathscr{S}$ is the universal family of pairs (X, S) induced by it, there is a universal embedded resolution

$$\begin{tikzcd}
\mathscr{C} \arrow[r, hook] \arrow[d, "\rho"] & \tilde{\mathscr{S}} \arrow[d, "\varepsilon"] \\
\mathscr{X} \arrow[r, hook] \arrow[d] & \mathscr{S} \arrow[dl] \\
V_{g,\delta} &
\end{tikzcd}$$

where ε is the blow-up of $\mathscr{S}$ along the nodal locus $\mathcal{N}_\delta \subset \mathscr{X} \hookrightarrow \mathscr{S}$ and ρ is fiber-wise over $V_{g,\delta}$ the normalization. Every locally trivial first order deformations of a pair (X, S) induce one of the pair $(C, \tilde{S})$ parametrized by $H^1(\tilde{S}, \mathscr{F}_C)$.

Proof of Theorem 2.4. First of all, notice that we can assume $l = 0$. By Definition 2.3, the diagram

$$\begin{tikzcd}
\mathcal{V}_{g,\delta,l} \arrow[r] \arrow[d, "c_{g,\delta,l}"] & \mathcal{V}_{g,\delta} \arrow[d, "c"] \\
\mathcal{M}_{g-\delta,[2\delta]+l} \arrow[r, "\pi"] & \mathcal{M}_{g-\delta,[2\delta]}
\end{tikzcd}$$

is cartesian, where π is the map that forgets the last l marked points. The forgetful map π is surjective, therefore dominance of $c_{g,\delta,l}$ is equivalent to dominance of c.

We will prove that the differential of c is onto in the assumed range. Indeed,

$$dc : H^1(\tilde{S}, \mathscr{F}_C(-E)) \to H^1(C, T_C(-p_1 - q_1 - \ldots - p_\delta - q_\delta))$$

And by Corollary 2.11, the isomorphisms (2.5) and the vanishing (2.2), we have

$$\operatorname{coker} dc \cong H^2(\tilde{S}, \varepsilon^* T_S(-C - E)).$$

We call H the polarization $\mathcal{O}_S(X)$. By Serre duality and since $\varepsilon^* H = C + 2E$,

$$H^2(\tilde{S}, \varepsilon^* T_S(-C - E)) \cong H^0(\tilde{S}, \varepsilon^* \Omega^1_S(H)).$$

But

$$R\varepsilon_*(\varepsilon^* \Omega^1_S(H)) \cong \Omega^1_S(H) \otimes R\varepsilon_* \mathcal{O}_{\tilde{S}}$$

and since ε is birational $R\varepsilon_* \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_S$, i.e., the higher direct images $R^i \varepsilon_* \mathcal{O}_{\tilde{S}}$ vanish for $i > 0$,

$$H^0(\tilde{S}, \varepsilon^* \Omega^1_S(H)) \cong H^0(S, \Omega^1_S(H))$$

and this last group is trivial when $g \leqslant 9$ or $g = 11$. When $g = 10$, $H^0(S, \Omega^1_S(H)) \cong \mathbb{C}$, meaning, the image is divisorial. See [Bea04, §5.2]. □

Corollary 2.14. *When* $g = 10$*, the image of* $c_{g,\delta,l}$ *has always codimension one and its class in* $Pic_{\mathbb{Q}}(\mathcal{M}_{10-\delta,[2\delta]+l})$ *is given by*

$$\left(c_{g,\delta,l}\right)_* [\mathcal{V}_{10,\delta,l}] = p^* \left(\frac{1}{2\delta!} (7\lambda + \psi)\right),$$

where $p : \mathcal{M}_{10,[2\delta]+l} \to \mathcal{M}_{10,[2\delta]}$ *is the map that forgets the* l*-marked points and* ψ *is the* $\mathbb{Z}_2^{\oplus\delta}$*-invariant class induced by* $\psi = [\psi_1 + \psi_2] + \ldots + [\psi_{2\delta-1} + \psi_{2\delta}]$.

Proof. To compute the class of this divisor in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{10-\delta,[2\delta]})$ we just need to pull back the class $\overline{\mathcal{K}}$ of the closure of the K3 locus in $\mathcal{M}_{10}$ by the boundary map $\xi : \overline{\mathcal{M}}_{10-\delta,2\delta} \to \overline{\mathcal{M}}_{10}$ and then push it down by the $\mathbb{Z}_2^{\oplus 2}$-quotient

$$\pi : \overline{\mathcal{M}}_{10-\delta,2\delta} \to \overline{\mathcal{M}}_{10-\delta,[2\delta]}.$$

If we call c the map $c_{g,\delta}$, then

$$c_*[\mathcal{V}_{10,\delta}] = \frac{1}{n!}\pi_* \left(\xi^*\overline{\mathcal{K}}\right).$$

If we restrict this to $\mathcal{M}_{10-\delta,[2\delta]}$, we have our result. □

Remark 2.15. Take for example $g = 10$ and $\delta = 1$. If C is a general genus 9 curve there is a codimension one cycle in the symmetric product without the diagonal $\Gamma \subset C^{[2]} \setminus \triangle$ consisting of points $p + q$ such that after identifying them, the nodal curve $C/_{p\sim q}$ lies on a K3. Let $C^{[2]} \setminus \triangle$ be a general fiber of $\pi : \overline{\mathcal{M}}_{9,[2]} \to \overline{\mathcal{M}}_9$, since the complex structure of the curve along the fiber is constant, the Hodge bundle restricts to the trivial bundle and the class of Γ in $\mathrm{Pic}(C^{[2]} \setminus \triangle)$ is given by

$$\frac{7\lambda + \psi_1 + \psi_2}{2} \cdot \pi^*(pt) = K_C + C.$$

The same argument works for $\delta \geqslant 1$.

2.3 Deformation theory of pointed nodal curves on K3 surfaces

The goal of this section is to show dominance of the map

$$\pi : \mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,\delta+l},$$

when $3\delta + l \leqslant g$. If we count dimensions naively, every marked point $y_1, \ldots, y_l \in S$ should impose one linear condition on the linear system $|H|$ and for an hyperplane section to be nodal at $x_1, \ldots, x_\delta$ we need to impose that it contains all tangent 2-planes at those points. Thus, every node should impose 3 linear conditions on $|H|$. If the conditions imposed are independent, the map π should be dominant when $3\delta + l \leqslant g$. The issue is to disregard cusps or higher singularities at $x_1, \ldots, x_\delta$.

Definition 2.16. Let S be a K3 surface, $X \subset S$ a (nodal) curve and $y_1, \dots, y_l \in X$ marked points away from the nodes. We define

$$T_S\langle X \mid y_1, \dots, y_l\rangle \subset T_S \otimes \mathcal{I}_{y_1+\ldots+y_l}$$

to be the inverse image of $T_X(-y_1 - \ldots - y_l) \subset (T_S \otimes \mathcal{I}_{y_1+\ldots+y_l})\,|_X$ under the natural restriction

$$T_S \otimes \mathcal{I}_{y_1+\ldots+y_l} \to (T_S \otimes \mathcal{I}_{y_1+\ldots+y_l})\,|_X\,.$$

The sheaf $T_S\langle X \mid y_1, \dots, y_l\rangle$ is called the *sheaf of germs of tangent vectors to the pointed K3* $(S, y_1, \dots, y_l)$ *which are tangent to the pointed* $(X, y_1, \dots, y_l)$.

To simplify the notation we write $y = y_1 + \ldots + y_l$. The sheaf

$$T_S\langle X \mid y_1, \dots, y_l\rangle$$

sits in two exact sequences coming from restriction and inclusion respectively:

$$\begin{aligned} 0 \to T_S(-X) \otimes \mathcal{I}_y \to T_S\langle X \mid y_1, \dots, y_l\rangle \to T_X(-y) \to 0, \\ 0 \to T_S\langle X \mid y_1, \dots, y_l\rangle \to T_S \otimes \mathcal{I}_y \to \mathcal{N}'_{X/S}(-y) \to 0. \end{aligned} \tag{2.7}$$

The following proposition can be found in [Ser06, §3.4.4] without the markings. The case of closed embeddings with markings is straightforward to extend, but for sake of completeness we give a proof.

Proposition 2.17. *Locally trivial first order deformations of the pointed closed embedding*

$$(y_1, \dots, y_l \in X \hookrightarrow S)$$

are parametrized by $H^1(S, T_S\langle X \mid y_1, \dots, y_l\rangle)$. *The spaces* H^0 *and* H^2 *of the same sheaf parametrize local automorphisms an obstructions respectively and they both vanish when* $g \geqslant 3$ *and* $\delta \leqslant g - 2$.

Proof. Let $\mathscr{X} \hookrightarrow \mathscr{S}$ be a locally trivial first order deformation of $X \hookrightarrow S$ and

$$\tilde{y}_1, \dots \tilde{y}_l : \mathbb{I} \to \mathscr{X}$$

the sections corresponding to the markings. Here $\mathbb{I}$ denotes the scheme Spec $(\mathbb{C}[\varepsilon]/\varepsilon^2)$. Let $\{U_i\}_{i \in I}$ be an affine open cover of S,

$$\{V_i = X \cap U_i\}_{i \in I}$$

the induced affine open cover of X and $\{W_i = (\cup_k \{y_k\}) \cap U_i\}$ the induced cover of the zero dimensional scheme $y = y_1 + \ldots + y_l \subset X$. Notice that some W_i might be empty. As usual, every locally trivial first order deformation is obtained by gluing the trivial deformations

$$W_i \times \mathbb{I} \hookrightarrow V_i \times \mathbb{I} \hookrightarrow U_i \times \mathbb{I}$$

along the intersections $W_{ij} \times \mathbb{I}$, $V_{ij} \times \mathbb{I}$ and $U_{ij} \times \mathbb{I}$. This is equivalent to giving local automorphisms (A_{ij}, B_{ij}, C_{ij}) such that the following diagram commutes:

$$\begin{array}{ccccc} W_{ij} \times \mathbb{I} & \hookrightarrow & V_{ij} \times \mathbb{I} & \hookrightarrow & U_{ij} \times \mathbb{I} \\ \downarrow{\scriptstyle C_{ij}} & & \downarrow{\scriptstyle B_{ij}} & & \downarrow{\scriptstyle A_{ij}} \\ W_{ij} \times \mathbb{I} & \hookrightarrow & V_{ij} \times \mathbb{I} & \hookrightarrow & U_{ij} \times \mathbb{I}. \end{array}$$

This correspond to sections $D_{ij} \in \Gamma(U_{ij}, T_S \otimes \mathcal{I}_y)$, $d_{ij} \in \Gamma(V_ij, T_X \otimes \mathcal{I}_y)$ such that $D_{ij} |_X = d_{ij}$, that is, sections D_{ij} of $T_S\langle X \mid y_1, \ldots, y_l\rangle$. To check the cocycle condition and obstruction space is the same as without the markings, we refer to [Ser06, Prop. 1.2.9 and Prop. 1.2.12] for details. The vanishing of H^0 follows from the vanishing of $H^0(T_S) = 0$ and the inclusion in the second exact sequence of (2.7). The vanishing of H^2 follows from the inclusion

$$T_S\langle X \mid y_1, \ldots, y_l\rangle \subset T_S\langle X\rangle.$$

The cokernel of this inclusion is supported on the points $y_1, \ldots, y_l$ and $H^2(T_S\langle X\rangle)$ vanishes in the established range, cf. [FKGS08, Prop. 4.8]. □

We can give an alternative proof for the dominance of the normalization map. Let

$$(S, X, x_1, \ldots, x_\delta, y_1, \ldots, y_l) \in \mathcal{V}_{g,\delta,l}$$

be a general point and $f : C \to X$ the normalization. We call

$$p + q = p_1 + q_1 + \ldots + p_\delta + q_\delta$$

the preimage of the nodes and $y = y_1 + \ldots + y_l$ the marked points.

Proposition 2.18. *The differential of the normalization map*

$$dc_{g,\delta,l} : \mathcal{V}_{g,\delta,l} \to \mathcal{M}_{g-\delta,[2\delta]+l}$$

can be identified with

$$H^1(S, T_S\langle X \mid y_1, \ldots, y_l\rangle) \to H^1(X, T_X(-y)).$$

The identification on the right is given by the isomorphism induced by f_**;*

$$H^1(C, T_C(-(p+q) - y)) \xrightarrow{\sim} H^1(X, f_*(T_C(-(p+q) - y))) \cong H^1(X, T_X(-y)).$$

Proof. From the proof of Proposition 2.17, one can see that the map induced by restriction sends locally trivial first order deformations of (S, X, y) to locally trivial first order deformations of (X, y). By Lemma 2.12, there is a natural isomorphism

$$f_* T_C(-(p+q)) \cong T_X.$$

After tensoring with $\mathcal{O}(-y)$, the composition with the natural isomorphism

$$H^1(C, T_C(-(p+q)-y)) \cong H^1(X, f_*T_C(-(p+q)-y)),$$

identifies first order deformations of (X, y) preserving the nodes with deformations of (C, y) together with the marked points $p+q$. $\square$

As corollary we have:

Corollary 2.19. *The normalization map $c_{g,\delta,l}$ is dominant for $g \geqslant 3$, $\delta \leqslant g-2$, and $g \leqslant 11$ with $g \neq 10$.*

Proof. By the first exact sequence (2.7), and Prop. 2.17, the cokernel of $dc_{g,\delta,l}$ is isomorphic to

$$H^2(S, T_S(-X) \otimes \mathcal{I}_y).$$

On the other hand, the cokernel of the inclusion $T_S(-X) \otimes \mathcal{I}_y \subset T_S(-X)$ is supported on the points y. Thus,

$$H^2(S, T_S(-X) \otimes \mathcal{I}_y) \cong H^2(S, T_S(-X)) \cong H^0(S, \Omega^1_S(X)).$$

The dimension of the last vector space for general $(S, \mathcal{O}_S(X)) \in \mathcal{F}_g$ is zero when $g \neq 10$ and $g \leqslant 11$, cf. [Bea04, §5.2]. $\square$

Recall that if $f : C \to X \subset S$ is the normalization, and $\mathcal{N}_f$ is the normal sheaf of f sitting in the exact sequence

$$0 \to T_C \to f^*T_S \to N_f \to 0,$$

then (cf. [FKGS08, Lemma 4.16]),

$$f_*N_f \cong \mathcal{N}'_{X/S} \tag{2.8}$$

and from the exact sequence above, since S is a K3 surface, $N_f \cong \omega_C$. Thus, $h^1(\mathcal{N}'_{X/S}) = 1$ and the kernel of the surjection

$$\lambda : H^1(S, T_S) \to H^1(X, \mathcal{N}'_{X/S})$$

is 19 dimensional. The kernel of λ can be thought as the tangent space of $\mathcal{F}_g$ at $(S, \mathcal{O}_S(X))$ and the short exact sequence

$$0 \to H^0(\mathcal{N}_{X/C}) \to H^1(T_S\langle X\rangle) \to \ker(\lambda) \to 0$$

as the natural differential sequence

$$0 \to T_{[X]}\left(V_\delta(S, H)\right) \to T_{(S,X)}\mathcal{V}_{g,\delta} \to T_{(S,H)}\mathcal{F}_g \to 0.$$

Here $H = \mathcal{O}_S(X)$, see [FKGS08, Prop. 4.8]. Let $\varepsilon : \tilde{S} \to S$ be the blow-up of S at the marked points $x_1, \dots, x_\delta, y_1, \dots, y_l$ and

$$E = E_1 + \ldots + E_\delta, \quad F = F_1 + \ldots + F_l$$

the exceptional divisors corresponding to the nodes and marked points respectively. The proper transform C of X lies in $|\varepsilon^* H - 2E - F|$ and the following sequence is exact:

$$0 \to \mathscr{F}_C(-E-F) \to \varepsilon^* T_S(-E-F) \to N_f(-(p+q)-y) \to 0. \tag{2.9}$$

Lemma 2.20. *In the same setting as above,*

$$\varepsilon_* \left(\mathscr{F}_C(-F)\right) \cong T_S\langle X, y_1, \dots, y_l\rangle \text{ and } H^1(\mathscr{F}_C(-E-F)) \cong H^1(\mathscr{F}_C(-F)).$$

Proof. There are maps as in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & T_S\langle X, y_1, \dots, y_l\rangle \arrow[r] \arrow[d] & T_S \otimes \mathcal{I}_y \arrow[r] \arrow[d, "\wr"] & \mathcal{N}'_{X/S}(-y) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varepsilon_* \left(\mathscr{F}_C(-F)\right) \arrow[r] & \varepsilon_* \left(\varepsilon^* T_S(-F)\right) \arrow[r] & \mathcal{N}'_{X/S}(-y) \arrow[r] & 0.
\end{tikzcd}$$

The map in the middle comes from the isomorphism $\varepsilon_* \left(\mathcal{O}(-F)\right) \cong \varepsilon_* \varepsilon^* \mathcal{I}_y$. This induces an isomorphism

$$T_S\langle X, y_1, \dots, y_l\rangle \cong \varepsilon_* \left(\mathscr{F}_C(-F)\right).$$

The second assertion is equivalent to Proposition 2.10, replacing $\mathscr{F}_C$ with $\mathscr{F}_C(-F)$. ☐

Now consider the diagram coming from (2.9):

$$\begin{tikzcd}
H^1\left(\tilde{S}, \mathscr{F}_C(-E-F)\right) \arrow[r] & H^1\left(\tilde{S}, \varepsilon^* T_S(-E-F)\right) \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & H^1\left(C, N_f(-(p+q)-y)\right) \arrow[d] \\
 & & H^1\left(C, N_f\right).
\end{tikzcd} \tag{2.10}$$

Proposition 2.21. *The tangent space of $\mathcal{F}_{g,\delta+l}$ at $(S, H, x_1, \dots, x_\delta, y_1, \dots, y_l)$ can be identified with* $\ker(\beta)$ *and the differential of the map $\pi : \mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,\delta+l}$ is given by*

$$d\pi : H^1\left(S, \mathscr{F}_C(-E-F)\right) \to \ker(\alpha) \subset \ker(\beta).$$

In particular, π is dominant if for some point $(S, X, x_1, \dots, x_\delta, y_1, \dots, y_l)$ the map induced by inclusion

$$H^1\left(C, N_f(-(p+q)-y)\right) \to H^1\left(C, N_f\right)$$

is an isomorphism, and generically finite onto its image if

$$H^0\left(C, N_f(-(p+q)-y)\right) = 0.$$

As corollary we have Theorem 2.5:

Proof of Thm 2.5. Recall that in the established range, the map

$$\mathcal{V}_{g,\delta,l} \to \mathcal{M}_{g-\delta,[2\delta]+l}$$

is dominant. In particular, for a general point

$$(S, X, x_1, \ldots, x_\delta, y_1, \ldots, y_l) \in \mathcal{V}_{g,\delta,l},$$

the points lying over the nodes and marked points

$$p + q = (p_1 + q_1) + \ldots + (p_\delta + q_\delta), \quad y = y_1 + \ldots y_l$$

are general in the symmetric product $\mathrm{Sym}^{2\delta+l}C$. Since S is a K3 surface, $N_f \cong \omega_C$ and the surjection

$$H^1\left(C, N_f\left(-\sum \tilde{y}_i - E\,|_C\right)\right) \to H^1\left(C, N_f\left(-\sum \tilde{y}_i\right)\right)$$

is an isomorphism if and only if $h^0(C, \mathcal{O}_C((p+q)+y)) \leqslant 1$. This holds when $2\delta + l \leqslant g - \delta$. Now assume that $3\delta + l = g$. We will show that the general fiber of $\mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,\delta+l}$ is a point. Let $(S, H, x_1, \ldots, x_\delta, y_1, \ldots, y_l)$ be a general point in $\mathcal{F}_{g,\delta,l}$ and $\varepsilon : \tilde{S} \to S$ the blow up along the marked points, with

$$E = E_1 + \ldots + E_\delta, \;\; F = F_1 + \ldots + F_l$$

the exceptional divisors corresponding to $x = x_1 + \ldots + x_\delta$ and $y = y_1 + \ldots + y_l$ respectively. Since

$$R^i \varepsilon_* \mathcal{O}_{\tilde{S}}(-2E - F) = 0 \text{ for } i > 0,$$

the push forward give us an isomorphism

$$H^0\left(\tilde{S}, \mathcal{O}_{\tilde{S}}(-2E - F)\right) \tilde{\to} H^0(S, H \otimes \mathcal{I}_{2x+y}).$$

We are under the assumption that $3\delta + l = g \leqslant 11$ and $g \neq 10$. By dominance of $c_{g,\delta,l}$, there exists an smooth curve $C \in |\varepsilon^* H - 2E - F|$ intersecting E and F transversally at exactly two and one points respectively. Therefore, the general element of $|C - 2E - F|$ is smooth and intersects E and F transversally. After pushing forward, the general element of

$$\mathbb{P}\left(H^0(S, H \otimes \mathcal{I}_{2x+y})\right)$$

is a nodal curve at x passing through y. By dimension count there are finitely many of them and the linear subspace $\mathbb{P}\left(H^0(S, H \otimes \mathcal{I}_{2x+y})\right) \subset |H|$ consist of a single element. $\square$

Now we prove the main proposition.

Proof of Proposition 2.21. We call $x = x_1 + \ldots + x_\delta$ and $y = y_1 + \ldots + y_l$ the nodes and marked points on S. The map $\varepsilon : \tilde{S} \to S$ is birational and finite when restricted to C. Thus,

$$R^i\varepsilon_*\mathcal{O}_{\tilde{S}} \cong R^i\varepsilon_*\mathcal{O}(-E-F) \cong 0 \text{ for } i > 0,$$

and there is an isomorphism coming from the Leray spectral sequence sending isomorphically the right triangle in (2.10) to

$$\begin{tikzcd} H^1\left(S, T_S \otimes \mathcal{I}_{x+y}\right) \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & H^1\left(X, \mathcal{N}'_{X/S} \otimes \mathcal{I}_{x+y}\right) \arrow[d] \\ & H^1\left(X, \mathcal{N}'_{X/S}\right). \end{tikzcd}$$

The map β is the composition

$$\beta : H^1(S, T_S \otimes \mathcal{I}_{x+y}) \to H^1(S, T_S) \to H^1(\mathcal{N}'_{X/S}).$$

Thus, elements of $\ker(\beta)$ can be interpreted as first order deformations of the pointed surface $(S, x_1, \ldots, x_\delta, y_1, \ldots, y_l)$ such that after forgetting the marked points, they lie in the kernel of $H^1(T_S) \to H^1(\mathcal{N}'_{X/S})$, that is, deformations that preserve the genus g marking. This proves the first assertion.

As showed in Proposition 2.10, the restriction of $\mathscr{F}_C$ to E is isomorphic to $\mathcal{O}_E(-1)^{\oplus 2}$. From the exact sequence

$$0 \to \mathscr{F}_C(-E-F) \to \mathscr{F}_C(-F) \to \mathcal{O}_E(-1)^{\oplus 2} \to 0,$$

one deduces that $R^i\varepsilon_*\mathscr{F}_C(-E-F) = R^i\mathcal{O}_{\tilde{S}}(-E-F) = 0$ for $i > 0$. The exact sequence (2.9) induces natural isomorphisms coming from the Leray spectral sequence

$$\begin{tikzcd} H^1(\tilde{S}, \mathscr{F}_C(-E-F)) \arrow[r] \arrow[d, "\cong"] & H^1(\tilde{S}, \varepsilon^*T_S(-E-F)) \arrow[d, "\cong"] \\ H^1(S, \varepsilon_*(\mathscr{F}_C(-E-F))) \arrow[r] & H^1(S, T_S \otimes \mathcal{I}_{x+y}). \end{tikzcd}$$

On the other hand, by Lemma 2.20, one has the following isomorphisms

$$\begin{tikzcd} H^1(\tilde{S}, \mathscr{F}_C(-E-F)) \arrow[r, "\cong"] \arrow[d, "\cong"] & H^1(\tilde{S}, \mathscr{F}_C(-F)) \arrow[d, "\cong"] \\ H^1(S, \varepsilon_*(\mathscr{F}_C(-E-F))) \arrow[r, "\cong"] & H^1(S, T_S\langle X, y_1, \ldots, y_l\rangle). \end{tikzcd}$$

Therefore the induced isomorphism

$$H^1(\tilde{S}, \mathscr{F}_C(-E-F)) \to H^1(S, T_S\langle X, y_1, \ldots, y_l\rangle)$$

identifies the tangent space of $\mathcal{V}_{g,\delta,l}$ with the vector space on the left. Moreover, the map

$$H^1\left(\tilde{S}, \mathscr{F}_C(-E-F)\right) \to H^1(\tilde{S}, \varepsilon^* T_S(-E-F)),$$

after the Leray isomorphism compose with the identification above, pushes down isomorphically to the tangent map

$$H^1\left(S, T_S\langle X, y_1, \ldots, y_l\rangle\right) \to H^1(S, T_S \otimes \mathcal{I}_{x+y}).$$

$\square$

Finally, we state a slight refinement of Proposition 2.21 that will become handy in the next chapter. Let r, s be non-negative integers such that $r \leqslant \delta$, and $s \leqslant l$. Let

$$\phi_{r,s} : \mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,r+s}$$

be the forgetful map that forgets the nodal curve, the last $\delta - r$ nodes and $l - s$ marked points, that is,

$$\phi_{r,s} : (S, X, x_1, \ldots, x_r, \ldots, x_\delta, y_1, \ldots, y_s, \ldots, y_l) \mapsto (S, \mathcal{O}(X), x_1, \ldots, x_r, y_1, \ldots, y_s).$$

This map is the composition of $\pi : \mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,\delta+l}$ with the forgetful map $\mathcal{F}_{g,\delta+l} \to \mathcal{F}_{g,r+s}$. Before going to the differential, let us fix some notation. As above, we call $\varepsilon : \tilde{S} \to S$ the blow-up of S at the marked points, C the proper transform of the nodal curve, E_i the exceptional divisor over the point x_i and F_i the one over y_i. Let

$$\begin{array}{rcl} E & = & E_1 + \ldots + E_r \\ E' & = & E_{r+1} + \ldots + E_\delta \end{array} \quad \text{and} \quad \begin{array}{rcl} F & = & F_1 + \ldots + F_s \\ F' & = & F_{r+1} + \ldots + F_l. \end{array}$$

As in (2.10), the diagram

$$\begin{tikzcd} H^1(\tilde{S}, \varepsilon^* T_S(-E-E'-F-F')) \arrow[r, "\beta_1"] \arrow[d] & H^1(C, N_f) \arrow[d, equal] \\ H^1\left(S, \varepsilon_* T_S(-E-F)\right) \arrow[r, "\beta_2"] & H^1(C, N_f) \end{tikzcd}$$

induces a surjection $\ker(\beta_1) \to \ker(\beta_2)$, corresponding to the differential of the forgetful map

$$T_{[S,x_1,\ldots,x_\delta,y_1,\ldots,y_l]}\mathcal{F}_{g,\delta+l} \to T_{[S,x_1,\ldots,x_r,y_1,\ldots,y_s]}\mathcal{F}_{g,r+s}.$$

Therefore, the differential of the composition $\mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,r+s}$ is given by the composition

$$H^1\left(\tilde{S}, \mathscr{F}_C(-E-E'-F-F')\right) \to \ker(\beta_1) \to \ker(\beta_2).$$

Proposition 2.22. *The differential of* $\mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,r+s}$ *at* $(S, X, x_1, \ldots, y_l)$ *is surjective if*

$$H^0\left(C, \mathcal{O}_C\left(\sum_1^r (p_i + q_i) + \sum_1^s y_j\right)\right) = 1$$

and injective if $s = l$ *and*

$$H^1\left(C, \mathcal{O}_C\left(\sum_1^r (p_i + q_i) + \sum_1^l y_j\right)\right) = 0.$$

Proof. To simplify the notation, let

$$\begin{aligned} p + q &= (p_1 + q_1) + \ldots + (p_r + q_r) \\ p' + q' &= (p_{r+1} + q_{r+1}) + \ldots + (p_\delta + q_\delta) \end{aligned} \quad \text{and} \quad \begin{aligned} y &= y_1 + \ldots + y_s \\ y' &= y_{s+1} + \ldots + y_l. \end{aligned}$$

Consider the diagram

$$\begin{tikzcd} H^1(\tilde{S}, \mathscr{F}_C(-E - E' - F - F')) \arrow[r] \arrow[d, "\gamma"] & H^1(\tilde{S}, \varepsilon^* T_S(-E - E' - F - F')) \arrow[r] \arrow[d] & H^1(C, N_f(-(p+q) - (p'+q') - y - y')) \arrow[d] \\ H^1(\tilde{S}, \mathscr{F}_C(-E - F)) \arrow[r] & H^1(S, \varepsilon^* T_S(-E - F)) \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & H^1(C, N_f(-(p+q) - y)) \arrow[d] \\ & & H^1(C, N_f). \end{tikzcd}$$

The vertical arrow in the middle is surjective and, the map induced by the composition,

$$H^1(\tilde{S}, \mathscr{F}_C(-E - E' - F - F')) \to H^1\left(S, \varepsilon^* T_S(-E - F)\right),$$

is surjective onto $\ker(\alpha)$. Since the tangent space of $\mathcal{F}_{g,r+s}$ is identified with $\ker(\beta)$, the differential is surjective if and only if the inclusion

$$\ker(\alpha) \subset \ker(\beta)$$

is an isomorphism. This holds when the map induced by inclusion

$$H^1\left(C, N_f(-(p+q) - y)\right) \to H^1\left(C, N_f\right)$$

is an isomorphism. Since $N_f \cong \omega_C$, this is equivalent to

$$H^0\left(C, \mathcal{O}_C((p+q) + y)\right) = 1.$$

For the second claim, notice that if $s < l$, the map contracts copies of the nodal curve X. If we assume $s = l$, then $F' = 0$ and, by the same argument as in Proposition 2.10,

$$\mathscr{F}_C(-E - F) \otimes \mathcal{O}_{E'} \cong \mathcal{O}_{E'}(-1)^{\oplus 2}.$$

In particular, the map γ is an isomorphism and injectivity is equivalent to

$$H^0(C, N_f(-(p+q) - y)) = 0.$$

By Serre's duality this is equivalent to the vanishing of $H^1(C, \mathcal{O}(p+q+y))$. $\square$

2.4 Comments on k-ampleness

Let L be a line bundle on a surface S. The line bundle L is said to be k-*very ample* if the restriction map

$$H^0(S, L) \to H^0(S, L \otimes \mathcal{O}_Z)$$

is surjective for every 0-dimensional scheme Z of length $h^0(\mathcal{O}_Z) = k + 1$. This notion generalizes the usual notion of very ample. For instance, 0-very ampleness is equivalent to being globally generated, and 1-very ampleness is equivalent to the usual definition of very ample. The standard geometrical interpretation of this property is the following: if $S^{[k+1]}$ is the Hilbert scheme of 0-dimensional subschemes of length $k + 1$, then, the map

$$S^{[k+1]} \to Gr(k + 1, H^0(L))$$

that sends $Z \in S^{[k+1]}$ to the quotient $H^0(L) \to H^0(L \otimes \mathcal{O}_Z)$, is a morphism if and only if S is k-very ample.

A point $(S, H, x_1, \ldots, x_\delta, y_1, \ldots, y_l) \in \mathcal{F}_{g,\delta+l}$ lies in the image of te map

$$\pi : \mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,\delta+l}$$

if

$$h^0(H \otimes \mathcal{I}_{2x_1+\ldots+2x_\delta+y_1+\ldots+y_l}) \geqslant 1$$

and, for a general section $s \in H^0(H \otimes \mathcal{I}_{2x_1+\ldots+2x_\delta+y_1+\ldots+y_l})$, the curve

$$\{s = 0\} \subset S$$

is nodal and irreducible. Notice that, if the points are pairwise distinct, then

$$h^0(\mathcal{O}_{2x_1+\ldots+2x_\delta+y_1+\ldots+y_l}) = 3\delta + l.$$

From the exact sequence

$$0 \to H \otimes \mathcal{I} \to H \to \mathcal{O}_{2x+y} \to 0,$$

$(3\delta + l - 1)$-very ampleness would imply

$$h^0(H \otimes \mathcal{I}_{2x+y}) = g + 1 - (3\delta + l),$$

where $x = x_1 + \ldots + x_\delta$ and $y = y_1 + \ldots + y_l$. The inequality $g \geqslant 3\delta + l$ is a necessary condition for the dominance of the map π. Knutsen in [Knu01] showed that, for a polarized K3 surface (S, H) of genus $g \geqslant 2$, the polarization H is k-very ample if and only if $H^2 \geqslant 4k$ and there is no effective divisor D on S such that

$$2D^2 \leqslant H \cdot D \leqslant D^2 + k + 1 \leqslant 2k + 2,$$

with equality at the first "$\leqslant$" if and only if $H \sim 2D$ and $H^2 \leqslant 4k+4$, and equality at the last "$\leqslant$" if and only if $H \sim 2D$ and $H^2 = 4k+4$. In particular, when $k = \frac{g-1}{2}$, and (S, H) is very general, $\mathrm{Pic}(S) = \mathbb{Z}H$ and $D = aH$. Plugging in D in the inequalities, one concludes that there is no such divisor, in particular, when $3\delta + l - 1 \leqslant \frac{g-1}{2}$, then

$$h^0(H \otimes \mathcal{I}_{2x+y}) \geqslant 1.$$

Question. When $3\delta + l - 1 \leqslant \frac{g-1}{2}$, is the general member of the linear system $\mathbb{P}(H^0(H \otimes \mathcal{I}_{2x+y}))$ nodal? For $(S, H, x_1, \dots, x_\delta, y_1, \dots, y_l) \in \mathcal{F}_{g,\delta+l}$ general in the locus where $h^0(H \otimes \mathcal{I}_{2x+y}) \geqslant 1$. Is the general element of the linear system nodal?

If the answer to the first question is yes, the moduli map

$$\mathcal{V}_{g,\delta,l} \to \mathcal{F}_{g,\delta+l}$$

would be dominant whenever

$$3\delta + l \leqslant \frac{g+1}{2}.$$

We strongly believe that this bound can be improved. We have proven that in the range $g \leqslant 11$ and $g \neq 10$, the bound $\frac{g+1}{2}$ is not close to be sharp, moreover when

$$3\delta + l \leqslant g,$$

then $h^0(H \otimes \mathcal{I}_{2x+y}) \geqslant 1$ and the general member of the linear system is nodal.

In general, if (X, L) is a polarized surface, $x_1, \dots, x_n \in X$ are distinct point on it and $m_1, \dots, m_n$ are multiplicities, one could ask if there is a curve in $|L|$ passing through the points x_i with multiplicities m_i. The scheme $\sum m_i x_i$ has length

$$h^0(\mathcal{O}_{m_1x_1+\ldots+m_nx_n}) = \sum_1^n \frac{m_i \cdot (m_i - 1)}{2}$$

and $(h^0 - 1)$-very ampleness of L implies the existence of such curve. The property of k-very ampleness is the one that guarantees the conditions of vanishing at x_i with multiplicity m_i to be independent conditions on the linear system $|L|$. This is certainly not always true, even if we assume the points to be general. For $\mathbb{P}^2$ there is the well known result ([Hir89]) stating that, general points p_i with multiplicities m_i fail to impose independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^2}(d)|$ when

$$\sum_1^n \frac{m_i \cdot (m_i + 1)}{2} \geqslant \lfloor \frac{(d+3)^2}{4} \rfloor.$$

Our case is simpler, since $m_i \in \{1, 2\}$. We don't need k-very ampleness, but some weaker condition on H. Moreover for (S, H) general, H is ample. We believe that we don't need to impose further conditions on H, as long as $3\delta + l \leqslant g$. By Proposition 2.22, to show this for arbitrary genus, one needs to find a point $(S, X, x, y) \in \mathcal{V}_{g,\delta,l}$, such that

$$H^0(C, \mathcal{O}_C((p+q)+y)) = 1, \tag{2.11}$$

where C is the normalization of X and $p + q$ is the pull-back of x under the normalization map. What is special in $g \leqslant 11$ and $g \neq 10$ is that, the normalized curve is general in $\mathcal{M}_{g-\delta}$ and the divisor $p + q + y$ is general in $\mathrm{Pic}^{2\delta+l}(C)$. It was already pointed out in the introduction of this chapter that curves lying on K3 surfaces capture properties of the general curve in various senses. It is plausible to expect that the same principle holds when the curves are nodal, in particular the inequality (2.11), for a general δ-nodal curve lying on a K3 surface. This motivates the following conjecture.

Conjecture. *With the same notation as above,*

- *for every* g, δ, l *satifying* $3\delta + l \leqslant g$, *there is a K3 surface* $(S, H) \in \mathcal{F}_g$ *and an* l*-pointed* δ*-nodal curve* $X \subset S$, *such that*

  $$H^0(C, \mathcal{O}_C(p+q+y)) = 1.$$

- *For every* g, δ, r, l *such that* $r \leqslant \delta$ *and* $2r + l \leqslant g - \delta$, *there is a K3 surface* $(S, H) \in \mathcal{F}_g$ *and an* l*-pointed* δ*-nodal curve* $X \subset S$, *such that*

  $$H^0(C, \mathcal{O}_C(p'+q'+y)) = 1,$$

  where $p' + q'$ *is the pull back of the first* r *nodes.*

From the fist conjecture would follow that the map π_1 in the diagram

$$\begin{tikzcd} \mathcal{V}_{g,\delta,l} \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathcal{F}_{g,\delta+l} \arrow[d] \\ & \mathcal{F}_{g,r+l} \end{tikzcd}$$

is always dominant and from the second conjecture would follow the dominance of π_2. Notice that, the difference in dimensions

$$\dim \mathcal{V}_{g,\delta,l} - \dim \mathcal{F}_{g,r+l} = g - (\delta + l + 2r)$$

is positive if $2r + l \leqslant g - \delta$. Notice also that when $r = \delta$, the fibers of π_1 are linear spaces. If the conjecture is true, then the map π_1 is would be generically finite of degree one when $3\delta + l = g$. We still don't know whether $\mathcal{V}_{g,\delta,l}$ is irreducible or not for $g \geqslant 12$. In any case, the conjecture would single out a component of $\mathcal{V}_{g,\delta,l}$ by the property of being birational to $\mathcal{F}_{g,\delta+l}$.

CHAPTER 3

The geometry of $\mathcal{F}_{11,n}$

3.1 Introduction

The question that concerns this chapter is about the birational geometry of the moduli space of polarized K3 surfaces of genus g, with marked points. The fist birational invariant that comes to mind is the Kodaira dimension defined in section §1.6. One should think the Kodaira dimension as an invariant that measures the complexity of the variety in question.

Question 3.1. What is the Kodaira dimension of $\mathcal{F}_{g,n}$?

Throughout this chapter $\mathrm{Kod}(\mathcal{F}_{g,n})$ stands for the Kodaira dimension of any smooth compactification of $\mathcal{F}_{g,n}$. As mentioned in the first chapter, it is a birational invariant between smooth projective varieties so, granted smoothnes, the choice of compactification is irrelevant.

We place the question in historical context. For $6 \leqslant g \leqslant 10$, Mukai [Muk88] constructed a rational variety $V_g \subset \mathbb{P}^{N_g}$, with $N_g = g-2+\dim V_g$, where the general K3 surface of genus g can be realized as g-linear section of V_g, that is,

$$S = V_g \cap \Lambda \ \text{ with } \ H = \mathcal{O}_S(1),$$

where $\Lambda \in \mathbb{G}(g, N_g)$, cf. Appendix §5.1. This produces a unirational parametrization of $\mathcal{F}_g$

$$\begin{array}{ccc} \mathbb{G}(g, N_g) & \dashrightarrow & \mathcal{F}_g \\ \Lambda & \mapsto & (V_g \cap \Lambda, \mathcal{O}(1)) \,. \end{array}$$

Moreover, any $(g+1)$-tuple of general points on V_g spanns a g-linear section Λ. This induces a unirational parametrization of $\mathcal{F}_{g,g+1}$

$$(V_g)^{\times g+1} \dashrightarrow \mathcal{F}_{g,g+1},$$

that sends a general tuple $(x_1, \dots, x_{g+1}) \in V_g^{\times g+1}$ to

$$(V_g \cap \mathrm{Spann}(x_1, \dots, x_{g+1}), \mathcal{O}(1), x_1, \dots, x_{g+1}) \in \mathcal{F}_{g,g+1}.$$

For $g = 2, 3, 4$ and 5 the analysis can be done explicitly, since a general K3 surface is a complete intersection on certain (weighted) projective space and for $g = 12, 13, 16, 18, 20$ Mukai also established structure theorems for the moduli space $\mathcal{F}_g$. It follows from Mukai's work the following theorem.

Theorem 3.2 (cf. [Muk88], [Muk92b], [Muk06] and [Muk12])**.** *The moduli space $\mathcal{F}_{g,n}$ is uniruled for*

- $g \leqslant 10$ *and* $n \leqslant g + 1$*, or*
- $n = 0$ *and* $g = 12, 13, 16, 18, 20$*.*

On the non-rationality side, Kondo [Kon93] sowed that when $g = p^2 + 1$, where p is a sufficiently large prime number, the moduli space $\mathcal{F}_g$ is of general type and later Gritsenko, Hulek and Sankaran [GHS07] showed that $\mathcal{F}_g$ is of general type for all $g > 62$ and $g = 47, 51, 55, 59, 61$. Both results are of transcendental nature and use the theory of toroidal compactifications of arithmetic quotients of bounded symmetric domains developed in [AMRT75].

The next major result regarding the birational geometry of $\mathcal{F}_g$ is due to Hassett [Has00]. Recall that, for a very general cubic fourfold $X \in |\mathcal{O}_{\mathbb{P}^5}|$, the lattice of integral $(2, 2)$ Hodge classes

$$A(X) = H^{2,2}(X) \cap H^4(X, \mathbb{Z})$$

is generated by a single element h^2, where h is the hyperplane class

$$h = c_1(\mathcal{O}_X(1)).$$

See [Vee86]. Hassett studied the loci of cubic fourfolds where this condition fails. More explicitly, let $\mathcal{C}_d$ be the locus of cubic fourfolds X that admit an embedding of a saturated rank two lattice

$$L = \langle h^2, [T] \rangle \hookrightarrow A(X)$$

with discriminant $\mathrm{disc}(L) = d$ and the surface T is algebraic not homologous to a complete intersection. He showed in [Has00] that $\mathcal{C}_d$ is an irreducible divisor in the moduli of cubic fourfolds and non-empty for $d > 6$ and $d \equiv 0, 2$ mod 6. Moreover, for $n \geqslant 2$ and $X \in \mathcal{C}_d$, when the discriminant satisfies the condition $d = 2(n^2 + n + 1)$, the Fano variety of lines

$$F(X) = \{\ell \in \mathbb{G}(1, 5) \mid \ell \subset X\}$$

is isomorphic the the Hilbert scheme $S^{[2]}$ of subschemes of length two on a polarized K3 surface (S, H) of degree $H^2 = d$. This induces a rational map

$$\mathcal{C}_d \dashrightarrow \mathcal{F}_{d/2+1}$$

that is birational when $d \equiv 2 \mod 6$. This map establishes a dictionary between K3 surfaces of genus $g = n^2 + n + 2$ and special cubic fourfolds. Nuer [Nue16] showed that $\mathcal{C}_{26}$ is unirational and therefore $\mathcal{F}_{14}$ is also unirational. Farkas and Verra [FV18] also used this dictionary tho show that $\mathcal{F}_{14,1}$ is rational.

Before we continue, let me remark that the forgetful map

$$u : \mathcal{F}_{g,n} \to \mathcal{F}_g$$

is a morphism fibered in Calabi-Yau varieties. By Iitaka's easy addition formula,

$$\mathrm{Kod}(\mathcal{F}_{g,n}) \leqslant \dim(\mathcal{F}_g) = 19.$$

In particular, $\mathcal{F}_{g,n}$ is never of general type for $n \geqslant 1$. Moreover, by [Kaw79],

$$\mathrm{Kod}(\mathcal{F}_{g,n+1}) \geqslant \mathrm{Kod}(\mathcal{F}_{g,n}).$$

When $\mathcal{F}_g$ is of general type, $\mathrm{Kod}(\mathcal{F}_{g,n}) = 19$.

3.1.1 Genus eleven.

The genus eleven case is particularly interesting. In Mukai's analysis, the description of the general K3 of genus eleven as the vanishing of a section of certain vector bundle over an homogenous projective variety is missing. As we already mention in Section §2.1, a general curve $[C] \in \mathcal{M}_{11}$ lies in a unique K3 surfaces as hyperplane section, cf. [Muk96]. For $n \leqslant 11$, this induces a rational map

$$\phi : \mathcal{M}_{11,n} \dashrightarrow \mathcal{F}_{11,n},$$

sending a general curve $[C, x_1, \ldots, x_n] \in \mathcal{M}_{11,n}$ to the associated K3 extension

$$(S, \mathcal{O}_S(C), x_1, \ldots, x_n) \in \mathcal{F}_{11,n}.$$

The fiber over a general point $(S, H, x_1, \ldots, x_n) \in \mathcal{F}_{11,n}$ is given by all hyperplane sections of $S \subset \mathbb{P}^{11}$ passing through the points $x_1, \ldots, x_n$. The general fiber is birational to

$$\mathbb{P}\left(H^0\left(S, H \otimes \mathcal{I}_{x_1+\ldots+x_n}\right)^{\vee}\right),$$

where $\mathcal{I}_{x_1+\ldots+x_n}$ is the ideal sheaf of the points. This map is dominant for $n \leqslant 11$ and birational for $n = 11$, moreover is birationally a $\mathbb{P}^{11-n}$-bundle over $\mathcal{F}_{11,n}$. Using this map and the unirationality of $\mathcal{M}_{11}$ (cf. [CR84]) one can conclude that $\mathcal{F}_{11}$ is unirational. On the other hand the Kodaira dimension of $\mathcal{M}_{11,11}$ is equal to 19, cf. [FV13, Thm 0.5] which implies

$$\mathrm{Kod}\left(\mathcal{F}_{11,11}\right) = 19.$$

Remark 3.3. It is claimed in [Log03, Table 3] that $\mathcal{M}_{11,n}$ is unirational for $n \leqslant 10$. Logan's argument only establishes the uniruledness of $\mathcal{M}_{11,n}$ by means of the Mukai map ψ. The birational description of the base is still missing when $n \leqslant 10$ and one of the motivations that inspired this work was to settle this issue.

The goal of this chapter is to answer the following question:

Question 3.4. What is the Kodaira dimension of $\mathcal{F}_{11,n}$ for $1 \leqslant n \leqslant 10$?

We were able to give partial answers described in the following theorems.

Theorem 3.5. *The moduli space* $\mathcal{F}_{11,n}$

- *is unirational for* $n \leqslant 6$*,*
- *is uniruled for* $n \leqslant 7$ *and*
- *it has non-negative Kodaira dimension for* $n \geqslant 9$*.*

Using the already mentioned fibration $\psi : \mathcal{M}_{11,n} \dashrightarrow \mathcal{F}_{11,n}$, we obtain the following corollary.

Corollary 3.6. *The moduli space* $\mathcal{M}_{11,n}$

- *is unirational for* $n \leqslant 6$ *and*
- *is not unirational for* $n = 9, 10$*.*

In the previous chapter we defined the moduli space $\mathcal{V}_{g,\delta,l}$ and showed that the moduli map induced by normalization

$$\mathcal{V}_{g,\delta,l} \to \mathcal{M}_{g-\delta,[2\delta]+l}$$

is dominant for small g, cf. Theorem 2.4. In particular, this map is dominant when $g = 11$ and $\delta \leqslant 9$. In this chapter we will show that the map is not only dominant but birational, giving us the following birational model for the $\mathbb{Z}_2^{\oplus\delta}$-quotient of the moduli of small genus curves with marked points.

Theorem 3.7. *The space* $\mathcal{V}_{11,\delta,l}$ *is birational to* $\mathcal{M}_{11-\delta,[2\delta]+l}$*, for* $0 \leqslant \delta \leqslant 9$*.*

Using the geometry of the forgetful map

$$\mathcal{V}_{11,\delta,l} \to \mathcal{F}_{11,\delta+l}$$

established in Theorem 2.5, we have as corollary:

Corollary 3.8. *For* $\delta \leqslant 9$*, there is a rational map*

$$\mathcal{M}_{11-\delta,[2\delta]+l} \dashrightarrow \mathcal{F}_{11,\delta+l},$$

dominant for $3\delta + l \leqslant 11$ *and birational for* $3\delta + l = 11$*.*

This, together with the known results about the Kodaira dimension of $\mathcal{M}_{g,n}$ for small g and n, give us a way to compute $\mathrm{Kod}(\mathcal{F}_{11,n})$ and conclude Theorem 3.5.

3.2 Hilbert scheme and projective cone.

As explained in the introduction, we will make full use of the diagram

$$\begin{tikzcd} & \mathcal{V}_{11,\delta,l} \arrow[dl, "\pi"'] \arrow[dr, "c_{g,\delta,l}"] & \\ \mathcal{F}_{11,\delta+l} & & \mathcal{M}_{11,[2\delta]+l}. \end{tikzcd}$$

Recall that

$$\dim\left(\mathcal{V}_{g,\delta,l}\right) = 19 + l + (g-\delta)$$

and this equals the dimension of $\mathcal{M}_{g-\delta,[2\delta]+l}$, when $g = 11$. The moduli map

$$c_{11,\delta,l} : \mathcal{V}_{11,\delta,l} \to \mathcal{M}_{g-\delta,[2\delta]+l}$$

is generically finite for $\delta \leqslant 9$. We will show that the general fiber of $c_{11,\delta,l}$ is irreducible, giving us Theorem 3.7. Recall that the diagram

$$\begin{tikzcd} \mathcal{V}_{g,\delta,l} \arrow[r] \arrow[d, "c_{g,\delta,l}"] & \mathcal{V}_{g,\delta} \arrow[d, "c"] \\ \mathcal{M}_{g-\delta,[2\delta]+l} \arrow[r, "\pi"] & \mathcal{M}_{g-\delta,[2\delta]} \end{tikzcd}$$

is cartesian, cf. Definition 2.3. Showing irreducibility of the general fiber of the $c_{g,\delta,l}$ can be reduced to show irreducibility of the general fiber of c. As in the proof of Theorem 2.4, we may assume $l = 0$.

Claim. The general fiber of $c_{11,\delta,l}$ is irreducible.

The generalities of this section can be found in [CD12, §2.1]. Let $\mathbf{F}_g$ be the component of the Hilbert scheme whose general point parameterize primitive K3 surfaces in $\mathbb{P}^g$ of degree $2g-2$. The group $\mathrm{PGL}(g+1)$ acts on $\mathbf{F}_g$, the dimension is $19 + (g+1)^2 - 1$ and the quotient map

$$\mathbf{F}_g \to \overline{\mathcal{F}}_g$$

is a $\mathrm{PGL}(g+1)$-bundle over an open set of $\overline{\mathcal{F}}_g$. Here $\overline{\mathcal{F}}_g$ stands for the projective quotient. Let $\mathbf{Hilb}_g$ be the component of the Hilbert scheme of curves on $\mathbb{P}^g$ whose general point parameterizes canonical curves lying on an hyperplane of $\mathbb{P}^g$. The quotient

$$\mathbf{Hilb}_g \to \overline{\mathcal{M}}_g,$$

over an open set, it is a $(\mathbb{P}^g)^\vee \times \mathrm{PGL}(g)$-bundle. We also define $\mathbf{V}_g$ to be the component of the flag Hilbert scheme (see [Kle81]) whose general point consist of a pair (S, X) with S a general point on $\mathbf{F}_g$ and X an hyperplane section of S.

An open subset of $\mathbf{V}_g$ is a $\mathbb{P}^g$-bundle over $\mathbf{F}_g$. The natural forgetful maps sit in a diagram

$$\begin{tikzcd} & \mathbf{V}_g \arrow[dl] \arrow[dr, "p"] & \\ \mathbf{F}_g & & \mathbf{Hilb}_g. \end{tikzcd}$$

Our situation is slightly different from the one described in [CD12], since we need to keep track of the nodes. For $0 \leqslant \delta \leqslant g-2$, we define the incidence variety

$$\mathbf{V}_{g,\delta} \subset \mathbf{V}_g \times (\mathbb{P}^g)^\delta$$

to be the closure of the set of points $(S, X, x_1, \dots, x_\delta)$ with X irreducible δ-nodal with nodes at $x_1, \dots, x_\delta$. We denote by $\mathbf{Nod}_{g,\delta}$ the closure of the locally closed subset in $\mathbf{Hilb}_g \times (\mathbb{P}^g)^\delta$ consisting of irreducible δ-nodal curves X together with points $(x_1, \dots, x_\delta) \in (\mathbb{P}^g)^\delta$ such that X is nodal at $x_1, \dots, x_\delta$.

Proposition 3.9. *The scheme $\boldsymbol{Nod}_{g,\delta}$ is irreducible.*

Proof. The rational map induced by normalization

$$\begin{array}{rcl} \mathbf{Nod}_{g,\delta} & \dashrightarrow & \mathbf{Hilb}_{g-\delta} \\ (X, x_1, \dots, x_\delta) & \mapsto & C \end{array}$$

is dominant and the fiber over a general canonical curve $C \subset \mathbb{P}^{g-\delta-1}$ is, up to projective transformations, birationally

$$(\mathrm{Sym}^2 C)^{\times\delta}.$$

$\square$

Notice that $\dim \mathbf{Nod}_{g,\delta} = g^2 - 4g - 4 - \delta$. Now, consider the forgetful map

$$p_{g,\delta} : \mathbf{V}_{g,\delta} \to \mathbf{Nod}_{g,\delta}.$$

Notice that

$$\dim \mathbf{V}_{g,\delta} = g^2 + 3g + 19 - \delta,$$

since for $0 \leqslant \delta \leqslant g-2$, $\dim V_\delta(S, H) = \dim |H| - \delta$, see [Tan82].

Proposition 3.10 (§2.2 [CD12]). *The general fiber of the map $p_{g,\delta}$ is irreducible.*

The irreducibility of the fiber relies on two facts. The first was proven by [Pin74] and states that a smooth K3 surface $S \subset \mathbb{P}^g$ can flatly degenerate inside $\mathbb{P}^g$ to the projective cone over any hyperplane section $S_X \subset \mathbb{P}^g$. Moreover, (cf. [CD12, Lemma 2.3]) one can do this inside the fiber of $p_{g,\delta}$, see Figure 3.1. Thus, if S_X is the projective cone over the nodal curve $X \subset S$ inside $\mathbb{P}^g$, then $(S_X, X, x_1, \dots, x_\delta)$ lies in every irreducible component of the general fiber of $p_{g,\delta}$.

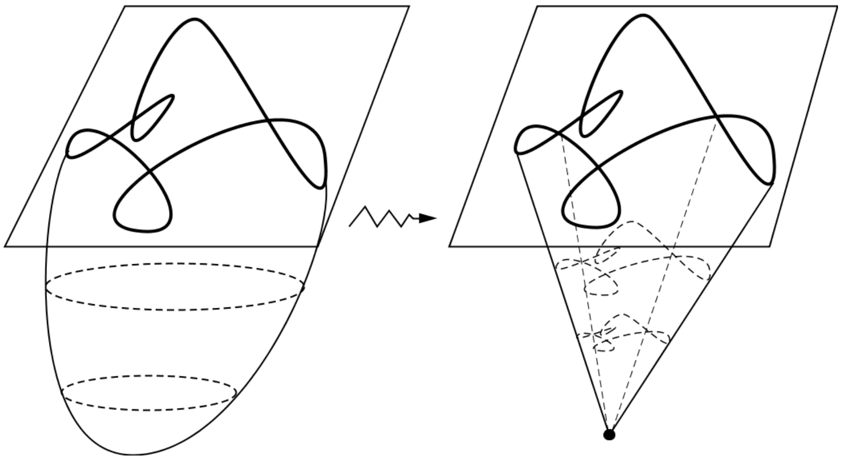

Figure 3.1: Flat degeneration from S to S_X in the fiber of $p_{g,\delta}$.

The second ingredient is the smoothness of the fiber at the point $(S_X, X, x_1, \dots, x_\delta)$. If so, then irreducibility of the general fiber $p_{g,\delta}$ follows.

The map

$$\begin{array}{rcl} \mathbf{V}_{g,\delta} & \to & \mathbf{V}_{g,\delta}/\Sigma_\delta \\ (S, X, x_1, \dots, x_\delta) & \mapsto & (S, X) \end{array}$$

is étale and the tangent space of the fiber at (S_X, X) (see [Ser06, §4.5.2] and [CD12, Lemma 2.4]) is isomorphic to

$$H^0(S_X, N_{S_X/\mathbb{P}^g}(-1)) \cong \bigoplus_{k \geqslant 1} H^0(X, N_{X/\mathbb{P}^g}(-k)).$$

The computation of these cohomology groups was done in [CM90, §3] by specializing to a *canonical graph curve*, that is, the union of $2g-2$ lines in $\mathbb{P}^{g-1}$, each meeting three other lines at distinct points. The dual graph of such a curve is a trivalent graph with $2g-2$ nodes and $3g-3$ edges as exemplified in Figure 3.2 for $g = 3$.

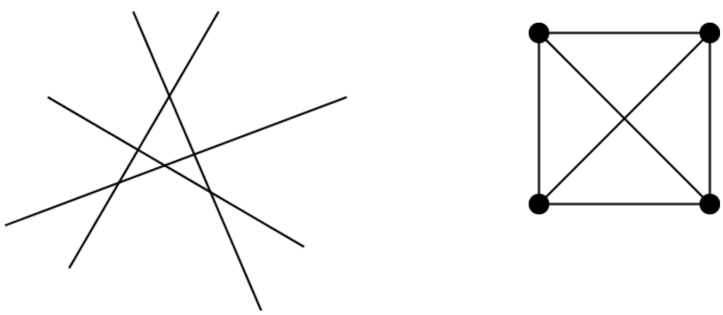

Figure 3.2: Canonical graph curve Γ_3 and its dual graph. This figure was taken from [BE91]

For such curve Γ_g, when $g = 11$

$$\bigoplus_{k\geqslant 1} H^0(\Gamma_{11}, N_{\Gamma_{11}/\mathbb{P}^{11}}(-k)) = H^0(\Gamma_{11}, N_{\Gamma_{11}/\mathbb{P}^{11}}(-1)) = 12.$$

Every irreducible δ-nodal curve degenerates to such a graph curve inside $\mathbb{P}^{g-1}$, see [CD12, Prop 2.6]. By upper semi-continuity of $h^0(S_X, N_{S_X/\mathbb{P}^g}(-1))$ one shows that the dimension of the general fiber of $p_{11,g}$ is exactly 12. Thus, the fiber is smooth at (S_X, X) and irreducibility follows.

We have proved Theorem 3.7. We summarize the proof.

Proof of Theorem 3.7. By Theorem 2.4 the map

$$c_{g,\delta} : \mathcal{V}_{11,\delta} \to \mathcal{M}_{11-\delta,[2\delta]}$$

is dominant when $1 \leqslant \delta \leqslant 9$, moreover it has zero dimensional general fiber. On the other hand $p_{11,\delta}$ is PGL$(g+1)$-equivariant and the diagram

$$\begin{tikzcd}
\mathbf{V}_{g,\delta} \arrow[r, "p_{11,\delta}"] \arrow[d, dashed] & \mathbf{Nod}_{g,\delta} \arrow[d, dashed] \\
\mathcal{V}_{11,\delta} \arrow[r, "c_{11,\delta}"] & \mathcal{M}_{11-\delta,[2\delta]}
\end{tikzcd}$$

commutes, where the vertical arrows are the corresponding PGL$(g+1)$ quotients. Recall that, the quotient separates orbit closures. The general fiber of $p_{11,\delta}$ is irreducible, thus the general fiber of $c_{11,\delta}$ is connected. Since is finite, consist of a single point. $\square$

3.3 Birational models for $\mathcal{F}_{11,n}$

We have constructed a birational map

$$c_{11,\delta,l} : \mathcal{V}_{11,\delta,l} \to \mathcal{M}_{11-\delta,[2\delta]+l}$$

for $\delta \leqslant 9$ and a dominant map

$$\mathcal{V}_{11,\delta,l} \to \mathcal{F}_{11,\delta+l},$$

for $3\delta + l \leqslant 11$. Moreover, the map is birational when $3\delta + l = 11$. In particular, when $\delta \leqslant 9$ and $3\delta + l \leqslant 11$, there is rational dominant map

$$\mathcal{M}_{11-\delta,2\delta+l} \dashrightarrow \mathcal{F}_{11,\delta+l},$$

birational when $3\delta + l = 11$.

Proof of Theorem 3.5. The moduli space $\mathcal{M}_{11-\delta,[2\delta]+l}$ is the $\mathbb{Z}_2^{\oplus\delta}$-quotient of $\mathcal{M}_{11-\delta,2\delta+l}$ and it is known [Log03, Thm 7.1] that $\mathcal{M}_{9,n}$ is unirational for $n \leqslant 8$ and therefore its quotient. In particular $\mathcal{V}_{11,2,4}$ is unirational and dominates $\mathcal{F}_{11,6}$. On the other hand $\mathcal{M}_{9,n}$ is uniruled for $n \leqslant 10$, cf. [FV13, Thm. 0.6]. Thus, the moduli space $\mathcal{V}_{11,2,5}$ is uniruled and the map $\mathcal{V}_{11,2,5} \to \mathcal{F}_{11,7}$ is birational.

Finally, $\mathcal{V}_{11,1,8} \to \mathcal{F}_{11,9}$ is birational giving us a birational map

$$\mathcal{M}_{10,10}/\mathbb{Z}_2 \overset{\sim}{\dashrightarrow} \mathcal{F}_{11,9}.$$

It is know (cf. [FV13, Thm. 0.1]) that the Kodaira dimension of the universal Jacobian over $\mathcal{M}_{10}$ has Kodaira dimension

$$\mathrm{Kod}\left(\overline{\mathscr{J}ac}^{10}_{10}\right) = 0.$$

There is a generically finite rational map

$$\overline{\mathcal{M}}_{10,10}/\mathbb{Z}_2 \to \overline{\mathcal{M}}_{10,10}/\Sigma_{10} \overset{\sim}{\dashrightarrow} \overline{\mathscr{J}ac}^{10}_{10},$$

where Σ_{10} is the symmetric group on 10 letters. Thus,

$$\mathrm{Kod}(\mathcal{F}_{11,9}) = \mathrm{Kod}(\overline{\mathcal{M}}_{10,10}/\mathbb{Z}_2) \geqslant 0.$$

□

3.4 Comments on $\mathcal{M}_{9,11}$, $\mathcal{M}_{9,12}$ and $\mathcal{M}_{10,10}$.

As mentioned in the first chapter §1.5.1, for $g > 3$, the moduli space $\overline{\mathcal{M}}_{g,n}$ is of general type for n large enough. In the range $4 \leqslant g \leqslant 11$, the tuples (g, n) for witch

$$\mathrm{Kod}(\overline{\mathcal{M}}_{g,n})$$

is unknown are just a few. These are:

$$(g, n) = (5, 14), (7, 14), (8, 13), (9, 11), (9, 12), \text{ and } (10, 10).$$

For the second and last tuple we have lower bounds (cf. [FV13]);

$$\mathrm{Kod}(\overline{\mathcal{M}}_{7,14}) \geqslant 0 \quad \text{and} \quad \mathrm{Kod}(\overline{\mathcal{M}}_{10,10}) \geqslant 0.$$

Regarding the middle cases, we don't even know the sign of the Kodaira dimension. This short section offers a possible path to provide lower bounds for the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ in the cases $(g, n) = (9, 11)$ and $(9, 12)$. We also discuss an upper bound for $(g, n) = (10, 10)$. The fact that we cannot say anything bout $\mathcal{F}_{11,8}$ is not a coincidence, the geometry of the moduli of genus eleven K3 surfaces with eight marked points is closely related with the geometry of $\mathcal{M}_{9,11}$ and $\mathcal{M}_{9,12}$.

Remark 3.11. In [FV13] is stated that $\mathrm{Kod}(\overline{\mathcal{M}}_{7,14}) \geqslant 2$, but we were not able to reconstruct the argument for such bound. On the other hand, one can check that the canonical class $K_{\overline{\mathcal{M}}_{7,12}}$ can be expressed as an effective combination of the pull back of the Brill-Noether divisor in $\overline{\mathcal{M}}_7$ corresponding to the closure of the locus of 4-gonal curves, the *symmetrization* of the divisor $\overline{\mathcal{D}}_7 \subset \overline{\mathcal{M}}_{7,7}$ constructed by Logan [Log03], corresponding to the closure of the locus of curves $[C, x_1, \dots, x_7]$ such that the divisor $x_1 + \dots + x_7$ moves on a pencil, and boundary divisors.

For any subset $S \subset \{1, \dots, 14\}$ of cardinality $|S| = 7$, there is a forgetful map

$$\pi_S : \overline{\mathcal{M}}_{7,14} \to \overline{\mathcal{M}}_{7,7}.$$

By "symmetrization" of $\overline{\mathcal{D}}_7$, we mean

$$\frac{1}{\binom{14}{7}} \sum_S \pi_S^* \overline{\mathcal{D}}_7.$$

3.4.1 Plugging in $\delta = 2$ and $l = 7$.

Consider the diagram

$$\begin{tikzcd} & \mathcal{V}_{11,2,7} \arrow[dl, "\pi_1"'] \arrow[dr] & \\ \mathcal{F}_{11,1+7} & & \mathcal{M}_{9,[4]+7}, \arrow[ll, dashed] \end{tikzcd}$$

Where π_1 is the map that forgets the nodal curve together with the first marked node. The map on the right is a birational isomorphism, giving us the dashed arrow that is generically finite as a consequence of the following lemma.

Lemma 3.12. *The map π_1 is generically finite of degree* 57.

Proof. By Proposition 2.22, the map π_1 is dominant if for a general point

$$(S, X, x_1, , x_2, y_1, \dots, y_7) \in \mathcal{V}_{11,2,7},$$

it holds that

$$H^0(C, \mathcal{O}_C(p_2 + q_2 + y_1 + \dots + y_7)) = 1,$$

where C is the normalization of X and $p_2 + q_2$ is the divisor lying over the node x_2. The curve C has genus 9 and the divisor

$$p_2 + q_2 + y_1 + \dots + y_7$$

is general in $\mathrm{Pic}^9(C)$, cf. Theorem 2.4. Comparing dimensions we have generic finiteness.

Let $\varepsilon : \tilde{S} \to S$ be the blow up of S at the marked points and $E + F_1 + \ldots + F_7$ the exceptional class. Let $P \subset |\varepsilon^* H - 2E - F_1 - \ldots - F_7|$ be a general pencil on the linear system. The base locus of P consist of

$$(\varepsilon^* H - 2E - F_1 - \ldots - F_7)^2 = 2g - 13 = 9$$

points. Consider the diagram induced by the pencil and its resolution

$$\begin{tikzcd} \mathscr{U} \arrow[r, "\mathrm{Bl}_9 \tilde{S}"] \arrow[d, "p"'] & \tilde{S} \arrow[dl, dashed] \\ P. & \end{tikzcd}$$

The degree of π_1 is equal to the number of singular fibers of p. By comparing the difference between the topological Euler characteristic of $\mathscr{U}$ and the product of the Euler characteristic of the base and the general fiber we have

$$\deg(\pi_1) = \chi(\mathscr{U}) - \chi(\mathbb{P}^1) \cdot \chi(p^{-1}(\text{point})).$$

The surface $\mathscr{U}$ is the blow-up of S at $8 + 9$ points, therefore $\chi(\mathscr{U}) = 41$ and

$$\deg(\pi_1) = 41 - (2 - 2(g-1)) \cdot 2 = 57.$$

□

We have the following corollary:

Corollary 3.13. *If the Kodaira dimension of $\mathcal{F}_{11,8}$ is non-negative, then*

$$\mathit{Kod}\left(\overline{\mathcal{M}}_{9,11}\right) \geqslant 0 \quad \mathit{and} \quad \mathrm{Kod}\left(\overline{\mathcal{M}}_{9,12}\right) \geqslant 25.$$

Proof. From the lemma above follows that, the rational map $\mathcal{M}_{9,[4]+7} \dashrightarrow \mathcal{F}_{11,8}$, is generically finite of degree 57. In particular, there is a generically finite rational map of degree 228 from $\mathcal{M}_{9,11}$ to $\mathcal{F}_{11,8}$. The lower bound for $\overline{\mathcal{M}}_{9,12}$ follows from the fact that the relative dualizing sheaf ω_π of the universal family $\pi : \overline{\mathcal{C}}_9 \to \overline{\mathcal{M}}_9$ is big, cf. [CHM97, §3]. The moduli space $\overline{\mathcal{M}}_{9,n}$ is birational to the n-th fiber product of $\overline{\mathcal{C}}_9$ over $\overline{\mathcal{M}}_9$. Since the relative dualizing sheaf ω_π and the sheaf of relative one forms Ω_π differ in codimension two,

$$K_{\overline{\mathcal{C}}_g^{\times 12}} = p^* K_{\overline{\mathcal{C}}_g^{\times 12}} \otimes q^* \omega_\pi, \tag{3.1}$$

where p and q sit in the cartesian diagram

$$\begin{tikzcd} \overline{\mathcal{C}}_9^{\times 12} \arrow[r, "q"] \arrow[d, "p"] & \overline{\mathcal{C}}_9 \arrow[d, "\pi"] \\ \overline{\mathcal{C}}_9^{\times 11} \arrow[r] & \overline{\mathcal{M}}_9. \end{tikzcd}$$

Since Kod $\left(\overline{\mathcal{M}}_{9,11}\right) \geqslant 0$, from (3.1),

$$\dim \left| mK_{\overline{\mathcal{C}}_g^{\times 12}} \right| \geqslant \dim |mq^*\omega_\pi| .$$

This gives us our result. $\square$

The question now is whether there is any reason to believe that

$$\mathrm{Kod}(\mathcal{F}_{11,8}) \geqslant 0.$$

Notice that, when $\delta = 1$ and $l = 7$, the map π on the diagram

$$\begin{tikzcd} & \mathcal{V}_{11,1,7} \arrow[dl, "\pi"'] \arrow[dr] & \\ \mathcal{F}_{11,8} & & \mathcal{M}_{10,[2]+7}, \arrow[ll, dashed, "F"] \end{tikzcd}$$

is birationally a $\mathbb{P}^1$-bundle. Thus, the rational map F contracts rational curves. The moduli space $\mathcal{M}_{10,[2]+7}$ is uniruled [FP05] as for $\mathcal{M}_{10,[2]+8}$, the Kodaira dimension is non-negative. It would be enough to show that through a general point of $\mathcal{M}_{10,[2]+7}$ passes a rational curve, but not a rational surface.

3.4.2 Comments on $\mathcal{M}_{10,10}$.

It seems to be the case that for $g \geqslant 2$, the moduli space $\overline{\mathcal{M}}_{g,n}$ has either negative Kodaira dimension (enjoys some rationality properties) or it is of general type. The only known example where this fails to be the case is $\overline{\mathcal{M}}_{11,11}$. As we already saw, this space is birational to $\mathcal{F}_{11,11}$ and both are 41 dimensional varieties of Kodaira dimension equals to 19. The goal of this short section is to discuss the existence of another example of intermediate Kodaira type.

Theorem 3.14. *The moduli space $\overline{\mathcal{M}}_{10,10}/\mathbb{Z}_2$ is of intermediate Kodaira type,*

$$0 \leqslant \mathrm{Kod}\left(\overline{\mathcal{M}}_{10,10}/\mathbb{Z}_2\right) \leqslant 19.$$

Proof. By Corollary 3.8, the moduli space $\mathcal{M}_{10,[2]+8}$ is birational to $\mathcal{F}_{11,9}$ and the last one admits a fibration

$$\mathcal{F}_{11,9} \to \mathcal{F}_{11}$$

with general fiber isomorphic to the 9-fold product of a K3 surface. By easy addition,

$$\mathrm{Kod}(\overline{\mathcal{M}}_{10,[2]+8}) \leqslant 19.$$

On the other hand, the full quotient

$$\overline{\mathcal{M}}_{10,[2]+8} \to \overline{\mathcal{M}}_{10,10}/\Sigma_{10}$$

is generically finite and the target has Kodaira dimension zero, cf. [FV13, Thm. 0.1]. $\square$

Notice that the quotient map

$$q : \overline{\mathcal{M}}_{10,10} \to \overline{\mathcal{M}}_{10,[2]+8}$$

is ramified over the boundary divisor $\delta_{0:\{1,2\}}$. Thus,

$$K_{\overline{\mathcal{M}}_{10,10}} = q^* K_{\overline{\mathcal{M}}_{10,[2]+8}} + \delta_{0:\{1,2\}}.$$

In order to have Theorem 3.14 for $\overline{\mathcal{M}}_{10,10}$, instead of the $\mathbb{Z}_2$-quotient, it would be enough to show that $m\delta_{0:\{1,2\}}$ lies in the fixed locus of the linear system

$$\left| m q^* K_{\overline{\mathcal{M}}_{10,[2]+8}} + m\delta_{0:\{1,2\}} \right|.$$

The divisor $\delta_{0:\{1,2\}}$ enjoys some negativity properties, for instance; it is extremal and rigid. Moreover, $\delta_{0:\{1,2\}}$ is anti-effective when restricted to itself. Probably, this can be used to show that $\overline{\mathcal{M}}_{10,10}$ is also of intermediate type. But we don't have a proof yet.

CHAPTER 4

Birational geometry of the strata

4.1 Introduction

As mentioned in the first chapter, the strata of differentials has gathered increasing attention from the algebraic geometry community. This chapter's main concern consist of giving a partial answer to the following question.

Question 4.1. How rational $\mathcal{H}_g^k(\mu)$?

As usual in moduli theory, the expectation is that for small genus at least some of the strata should be unirational, maybe even rational. We establish a non trivial range for g and partition μ, where the strata of holomorphic and quadratic differentials (i.e. $k = 1, 2$) has negative Kodaira dimension.

Theorem 4.2. *Let $\overline{\mathcal{H}}_g(\mu)$ be an irreducible stratum with length of partition $l(\mu)$. The birational geometry of $\overline{\mathcal{H}}_g(\mu)$ is summarized in the following table:*

Table 4.1

	Unirational	Uniruled
$3 \leqslant g \leqslant 6$	$l(\mu) \leqslant g-1$	No restriction on μ
$g = 7, 8$	?	No restriction on μ
$g = 9$	?	$l(\mu) \geqslant 7$
$g = 10$	?	$11 \leqslant l(\mu) < 18$
$g = 11$	?	$l(\mu) \geqslant 10$

The lack of dependency on the partition μ is surprising. As one can see, it only depends on the length of the partition, but not on the partition itself. The space that serves us as model and highlights the change from uniruledness to general type should be the moduli space of odd spin curves $\mathcal{S}_g^-$. Our work is inspired in [FV14].

The idea of the proof of Theorem 4.2 for uniruledness goes as follows; when the length of μ is grater than $g-1$ the forgetful map $\mathcal{H}_g(\mu) \to \mathcal{M}_g$ is dominant and, for small enough genus, the general smooth curve can be realized as a canonical curve given by a hyperplane section of a K3 surface $S \subset \mathbb{P}^g$. Since the divisor $\sum m_i x_i$ is canonical, it is the intersection of a codimension 2 plane with S. The rational curve is induced by the pencil spanned by this plane in $\mathbb{P}^g$.

The genus $g = 10$ is especially delicate and we treat it by studying 1-nodal models of geometric genus 10 curves lying on a K3 surface. We will make full use of our Theorem 2.4, that rephrased for $g = 11$ and $\delta = 1$, says that for a general genus 10 curves with two unordered marked points $[C, p+q] \in \mathcal{M}_{10,[2]}$, after identifying the points, the curve

$$X := C/p \sim q$$

has a unique K3 extension of genus eleven. As it will be seen in the proof, keeping track of the two unordered marked points is essential to our argument. It is not enough to use the fact that a general genus ten curve has a 1-nodal model lying on a K3, already proved in [FKGS08].

Recall that, for $g \geqslant 3$ and a holomorphic partition ν of $4g-4$, if at least one entry is odd, the space $\mathcal{Q}(\nu)$ is smooth and of pure dimension $2g-3+l(\nu)$, cf. [Lan02, Thm. 3 and 4], [Vee90] and [Sch16, Thm. 1.1]. In section §4.3.2, we turn to del Pezzo surfaces to construct rational curves on $\mathcal{Q}(\nu)$, obtaining the following result:

Theorem 4.3. *Let $\mathcal{Q}(\nu)$ be an irreducible stratum of holomorphic quadratic differentials. We assume ν to be a primitive partition of length $l(\nu)$. Then, for genus $3 \leqslant g \leqslant 6$ and $l(\nu) \geqslant g$, the moduli space $\mathcal{Q}(\nu)$ is uniruled.*

Regarding non-connected strata, we show the following:

Theorem 4.4. *For every genus $\mathcal{H}^{hyp}(2g-2)$ is unirational and even and odd strata are uniruled for any partition in the range $g \leqslant 8$. In genus $g = 9, 11$ the odd stratum $\mathcal{H}^-$ is uniruled for partition $\mu = (2, \dots, 2)$ and the Σ_{g-1}-quotient of the even stratum $\mathcal{H}^+$ for the same partition is uniruled in every genus.*

Remark 4.5. In [Lan02, Thm. 3 and 4], Lanneau provides an account of all the possibles connected components of the strata of quadratic differentials. In genus $g = 4$, this was found incomplete and improved by Chen and Möller [CM14]. In any case, merging these results together we have that, for homomorphic partition μ with at leas one odd entry (i.e., primitive) and length $l(\mu) \geqslant g$,

- the moduli space $\mathcal{Q}(\nu)$ is connected for $g = 3, 5$, and 6.

- For $g = 4$, the moduli space $\mathcal{Q}(\nu)$ is connected, unless $\nu = (3,3,3,3)$, where $\mathcal{Q}(\nu)$ breaks into three connected components;

$$\mathcal{Q}(3,3,3,3) = \mathcal{Q}^{hyp} \cup \mathcal{Q}^{irr} \cup \mathcal{Q}^{reg}.$$

  See [CM14, §7] for the precise description of each component.

In the proof of Theorem 4.3, we show that, when $3 \leqslant g \leqslant 6$ and ν is a primitive partition of length $l(\nu) \geqslant g$, every component of $\mathcal{Q}(\nu)$ dominating $\mathcal{M}_g$ is uniruled. In particular, regarding $g = 4$ case, we will have:

Theorem 4.6. *The connected component of the moduli space of quadratic differentials*

$$\mathcal{Q}^{reg}(3,3,3,3)$$

is uniruled and, the quotient of $\mathcal{Q}^{irr}(3,3,3,3)$ by the full symmetric group Σ_4, is also uniruled.

4.2 Unirationality for small genus

Irreducible strata

Recall that a variety is unirational if it is dominated by a rational variety. As above, $\mu = (m_1, \ldots, m_n)$ is a holomorphic partition of $2g-2$, with $g \geqslant 3$. We will assume the length of the partition partition to be at most $g-1$ and $\mathcal{H}_g(\mu)$ to be connected. A similar argument as in [FV14, Thm 3.1] can be used to obtain unirationality for $3 \leqslant g \leqslant 6$. The strategy is to construct a projective bundle $\mathcal{P}_g$ over a rational variety Σ that dominates $\overline{\mathcal{H}}_g(\mu)$, and use Proposition 1.26 to prove the following theorem:

Theorem 4.7. *Every connected strata $\mathcal{H}_g(\mu)$, with $3 \leqslant g \leqslant 6$ and $l(\mu) \leqslant g-1$ is unirational.*

For $3 \leqslant g \leqslant 6$, $\rho(g, 2, 6) \geqslant 0$. By choosing $[C, y_1, \ldots, y_{g-1}] \in \mathcal{M}_{g,g-1}$ and $A \in G^2_6(C)$ general, we can assume that the map $\phi_A : C \to \Gamma \subset \mathbb{P}^2$ realizes C as a $(10-g)$-nodal sextic and the marked points $y_1, \ldots, y_{g-1}$ are disjoint from the preimages of the nodes $\phi_A^{-1}(\mathrm{Sing}(\Gamma))$. Consider the following diagram of rational maps

$$\begin{tikzcd} & \mathcal{P}_g \arrow[dl, dashed, "\nu_g"'] \arrow[dr, dashed, "\pi_2"] & \\ \mathcal{M}_{g,n} & & \Sigma, \end{tikzcd} \tag{4.1}$$

where $\Sigma \subset |\mathcal{O}_{\mathbb{P}^2}(3)| \times (\mathbb{P}^2)^9$ is defined as

$$\Sigma := \left\{(E, x_1, \ldots, x_\delta, y_1, \ldots, y_{g-1}) \in |\mathcal{O}_{\mathbb{P}^2}(3)| \times (\mathbb{P}^2)^9 \mid x_1, \ldots, y_{g-1} \in E\right\}$$

and the fiber of π_2 over a general point $(E, \bar{x}, \bar{y}) \in \Sigma$ is the linear space of plane sextics Γ, nodal at $x_1, \ldots, x_\delta$ and with μ contact at the points $y_1, \ldots, y_{g-1}$ with the cubic E. In other words,

$$\pi_2^{-1}(E, \bar{x}, \bar{y}) := \left\{ \Gamma \in |\mathcal{O}_{\mathbb{P}^2}(6)| \;\middle|\; \begin{array}{l} \Gamma \text{ is nodal at } \bar{x} \text{ and} \\ \Gamma \cdot E = \sum_1^{g-1} m_i y_i + 2(x_1 + \ldots + x_\delta) \end{array} \right\}.$$

See Picture 4.1, for an example where E is a conic instead of cubic and $g = 4$. If the partition μ has length less than $g-1$, we complete it with zeros so that the length is $g-1$.

One can see that the map induced by the projection $\Sigma \to (\mathbb{P}^2)^9$ is birational. The map ν_g sends $(\Gamma, E, \bar{x}, \bar{y})$ to $[C, \tilde{y}]$, where C is the normalization of Γ and $\tilde{y}$ is obtained by omitting from $(y_1, \ldots, y_{g-1})$ the terms y_i where $m_i = 0$.

Proposition 4.8. *The map π_2 is dominant.*

Proof. Let $(E, \bar{x}, \bar{y})$ be a general point in Σ and $\varepsilon : X \to \mathbb{P}^2$ the blow up of $\mathbb{P}^2$ at $x_1 + \ldots + x_\delta$ with F the exceptional divisor and L the pull back of the line. Let $\mathcal{K}(\mu)$ be the kernel of the composition

$$\mathcal{O}_X(6L - 2F) \to \mathcal{O}_E(6L - 2F) \to \mathcal{O}_E(6L - 2F)\,|_{\sum m_i y_i}.$$

Since $E = 3L - F = -K_X$ and $4 \leqslant \delta \leqslant 7$, we obtain $h^1(-K_X) = 0$. From this one deduces that H^0 of the first map is surjective. If the general element of $\mathbb{P}(H^0(X, \mathcal{K}(\mu)))$ is nodal, then

$$\dim \pi_2^{-1}(E, \bar{x}, \bar{y}) = h^0(\mathcal{K}(\mu)) - 1 \geqslant 27 - 3\delta - 2(g-1) = 9 - \delta.$$

By specialization, its enough to show that the general element of the linear system $|\mathcal{K}(\mu)|$ is nodal, when the points coincide, that is, $y_1 = \ldots = y_{g-1} = y$. In this case, the fiber of π_2 consists of δ-nodal sextics, with nodes at $\bar{x}$ and intersecting the tangent line ℓ_y of E at y with order $2g-2$. This is a *generalized Severi variety*, and by [CH98, Prop. 2.1], the space of δ-nodal sextics with such prescribed tangency with a given line in non-empty, in particular $|\mathcal{K}(2g-2)|$ contains nodal curves and therefore, so it does $|\mathcal{K}(\mu)|$.

□

Now we can give a proof of unirationality in the established range.

Proof of Theorem 4.7. Let $3 \leqslant g \leqslant 6$ and $\delta = 10 - g$. We need to prove that ν_g dominates $\mathcal{H}_g(\mu)$. Let $(\Gamma, E, \bar{x}, \bar{y}) \in \mathcal{P}_g$, with $\Gamma \cdot E = \sum m_i y_i + 2(x_1 + \ldots + x_\delta)$ and $\nu : C \to \Gamma$ the normalization. Then

$$\mathcal{O}_C\left(\sum m_i y_i\right) \cong \nu^* \mathcal{O}_\Gamma(3)\left(-\nu^{-1}(\bar{x})\right)$$

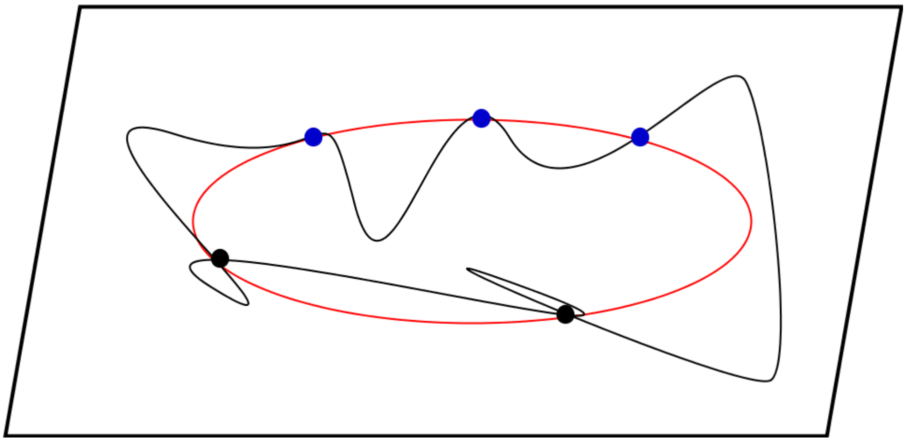

Figure 4.1: Unique conic through 5-points and 2-nodal curve with tangency $\mu = (3, 2, 1)$.

and by adjunction

$$\sum m_i y_i \sim K_C.$$

Thus, the image of ν_g lies in $\mathcal{H}_g(\mu)$. Now we prove dominance. Let $[C, \tilde{y}]$ be a general point in $\mathcal{H}_g(\mu)$. By assumption on g, we can choose $A \in G^2_6(C)$ general so that the associated map $\phi_A : C \to \mathbb{P}^2$ realize C as a δ-nodal curve and again by adjunction

$$h^0\left(C, \nu^* \mathcal{O}_\Gamma(3)\left(-\nu^{-1}(\bar{x}) - \sum_1^n m_i y_i\right)\right) = 1.$$

Thus, there exist a cubic $E \in |\mathcal{O}_{\mathbb{P}^2}(3)|$ such that

$$\Gamma \cdot E = \sum m_i y_i + 2(x_1 + \ldots + x_\delta).$$

By adding with multiplicity zero any $g - 1 - n$ points on E, we have that the image of

$$(\phi_A(C), x_1, \ldots, x_\delta, \bar{y}) \in \mathcal{P}_g$$

is exactly $[C, \tilde{y}] \in \mathcal{H}_g(\mu)$. □

This completes the first column in the table of Theorem 4.2.

4.2.1 Non-irreducible strata

Here we prove the unirationality of $\mathcal{H}_g^{hyp}(2g-2)$. Let $\mathscr{H}_{g,1} \subset \mathcal{M}_{g,1}$ be the space of pairs $[C, p]$, where C is an hyperelliptic curve and $p \in C$ is a Weierstrass point on it. Can be defined as the fiberproduct over $\mathcal{M}_g$ of the Weierstrass divisor $\mathscr{W}$ and the hyperelliptic locus $\mathrm{Hyp}_{g,1} \subset \mathcal{M}_{g,1}$;

$$\mathscr{H}_{g,1} = \mathrm{Hyp}_{g,1} \times_{\mathcal{M}_g} \mathscr{W}.$$

Recall that the space $\mathrm{Hyp}_g \subset \mathcal{M}_g$ is birrational to $\mathcal{M}_{0,2g+2}/\Sigma_{2g+2}$, where the map is the one that sends a smooth rational curve with divisor

$$x_1 + \ldots + x_{2g-2} \subset \mathbb{P}^1,$$

to the unique double cover ramified over that divisor:

$$\left(\mathbb{P}^1, x_1 + \ldots + x_{2g-2}\right) \mapsto \mathrm{Spec}\ \left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-(g-1))\right).$$

Here the algebra structure is given by the unique polynomial f of degree $2g-2$ with zeroes at the points $x_i's$,

$$f^{-1} : \mathcal{O}_{\mathbb{P}^1}(2g-2) \to \mathcal{O}_{\mathbb{P}^1}.$$

In particular there is a dominant map

$$\mathcal{M}_{0,2g+2} \to \mathscr{H}_{g,1}$$

sending the tuple of points $(x_1, \ldots, x_{2g+2})$ on $\mathbb{P}^1$ to the unique double cover

$$C \to \mathbb{P}^1$$

together with the Weierstrass point $f^*(x_1)_{red}$. This gives us unirationality of $\mathscr{H}_{g,1}$.

Lemma 4.9. *Let* C *be an hyperelliptic curve of genus* $g \geqslant 2$. *A point* $x \in C$ *is a Weierstrass point if and only if*

$$(2g-2)x \sim K_C.$$

Proof. If x is Weierstrass, then is a ramification point of the unique g^1_2, with associated cover $\pi : C \to \mathbb{P}^1$. By Riemann Hurwitz one concludes that

$$(2g-2)x \sim K_C \sim \pi^*\mathcal{O}(-2) + \mathcal{O}(R).$$

On the other hand if $(2g-2)x \sim K_C$,

$$\mathcal{O}_C \cong K_C(-(2g-2)x) \hookrightarrow K_C(-gx)$$

and $h^0(K_C - gx) \geqslant 1$. By Riemann-Roch we have that $h^0(gx) \geqslant 2$. □

As a consequence

$$\mathscr{H}_{g,1} = \mathcal{H}^{hyp}_g(2g-2)$$

and the first part of Theorem 4.4 follows.

4.3 Uniruledness results

As explained in the introduction, the general strategy is to construct pencils on K3 surfaces to prove uniruledness.

Lemma 4.10. *Let* $(S,H) \in \mathcal{F}_g$ *be a general polarized* K3 *of genus* $g \geqslant 2$ *and* $P \subset |H|$ *a general pencil whose general element is a smooth curve of genus* g*. Then the induced rational map* $P \dashrightarrow \mathcal{M}_g$ *is non trivial.*

Proof. As mentioned in the introduction, for general $(S,H) \in \mathcal{F}_g$ and $\delta \leqslant g$, the Severi variety of δ-nodal curves

$$V_\delta(S,H) = \{C \in |H| \mid C \text{ is } \delta\text{-nodal and irreducible}\}$$

is non-empty, regular and of codimension g in $|H|$. Choosing the pencil P general, it will intersect $V_1(S,H)$ non-trivially. Thus the induced map to $\mathcal{M}_g$ cannot be trivial. □

We need a slight refinement of the previous lemma.

Lemma 4.11. *Let* $(S,H) \in \mathcal{F}_g$ *be a general polarized* K3 *of genus* $g \geqslant 2$ *and* $P \subset |H|$ *any pencil whose general element is a smooth curve of genus* g*. Then the induced rational map* $P \dashrightarrow \mathcal{M}_g$ *is non trivial.*

Proof. By contradiction we assume the induced rational map to $\mathcal{M}_g$ is trivial. We blow-up S at the base locus of P:

$$\begin{tikzcd} \mathscr{U} \arrow[r, "\mathrm{Bl}_Z S"] \arrow[d, "\pi"'] & S \arrow[dl, dashed] \\ P. & \end{tikzcd} \tag{4.2}$$

Since the automorphism group of $\mathscr{U}$ is finite, up to an étale base change $B \to P$,

$$\tilde{\mathscr{U}} := \mathscr{U} \times_P B$$

is birational to $C \times B$, cf. [MM64, Thm. 2]. But $\mathscr{U}$ is simply connected, thus $\tilde{\mathscr{U}} \cong \mathscr{U}$ and $B \cong \mathbb{P}^1$. It follows that S is birational to $C \times \mathbb{P}^1$, but S is not ruled. □

Moreover, for $(S,H) \in \mathcal{F}_g$ general, every curve on a general pencil $P \subset |H|$ is irreducible and at worst nodal. The base locus of P consist of $2g-2$ points. If we resolve the map $S \dashrightarrow P$ by blowing up, we obtain a family $\mathscr{U}$ over P as in the proof of the lemma above. The general fiber F is a smooth genus g curve. From the relation

$$\chi(\mathscr{U},\mathbb{Z}) = \chi(P,\mathbb{Z}) \cdot \chi(F,\mathbb{Z}) + \text{number of singular fibers}$$

one deduces that on $\overline{\mathcal{M}}_g$, $P \cdot \delta_{\mathrm{irr}} = 6g+18$, where δ_{irr} is the divisor on $\overline{\mathcal{M}}_g$ whose general element correspond to a one nodal irreducible curve. In other words, the hypersurface parametrizing singular curves in the linear system $D_{S,H} \subset |H|$ has degree $6g+18$.

4.3.1 Uniruledness for $g \neq 10$.

Let $\mu = (m_1, \dots, m_n)$ be an holomorphic partition of $2g-2$ with length n and

$$[C, x_1, \dots, x_n] \in \mathcal{H}_g(\mu)$$

a point on the stratum. Assume $3 \leqslant g \leqslant 9$ or $g = 11$. The forgetful map $\pi : \mathcal{H}_g(\mu) \to \mathcal{M}_g$ is dominant when the length of the partition is greater or equal than $g-1$. See [Gen15]. The curve C is general and therefore can be embedded as a hyperplane section on a genus g polarized K3 surface $[S, H] \in \mathcal{F}_g$, cf. [Muk88] and [Muk96].

We will construct a rational curve

$$\mathbb{P}^1 \dashrightarrow \overline{\mathcal{H}}_g(\mu)$$

passing through $[C, x_1, \dots, x_n]$. Our curve is embedded in S as hyperplane section

$$C \cong S \cap H \hookrightarrow S \subset \mathbb{P}H^0(S, H)^{\vee} \cong \mathbb{P}^g.$$

See Figure 4.2, the blue curve represents C. The divisor

$$m_1x_1 + \ldots + m_nx_n \in \mathrm{Div}(C)$$

is canonical so it can be realized as a hyperplane section of $H \cong \mathbb{P}^{g-1}$, i.e. a point $\Lambda_\mu \in \mathbb{G}(g-2, \mathbb{P}^g)$ such that

$$\Lambda_\mu \cdot S = m_1x_1 + \ldots + m_nx_n.$$

The red line in Figure 4.2 represents Λ_μ. Let

$$P \cong \{H' \in (\mathbb{P}^g)^{\vee} \mid \Lambda_\mu \subset H'\}$$

be the pencil of hyperplanes containing Λ_μ in $\mathbb{P}^g$. Since $C \in P$ is smooth, for a general hyperplane $H' \in X$, the curve $C' = H' \cap S \hookrightarrow H' \cong \mathbb{P}^{g-1}$ is smooth and canonically embedded. Moreover, the hyperplane $\Lambda_\mu \subset \mathbb{P}^{g-1}$ is a canonical divisor of the form

$$\Lambda_\mu \cdot S = \Lambda_\mu \cdot C = m_1x_1 + \ldots + m_nx_n.$$

This construction gives us a map defined on an open subset of $P \cong \mathbb{P}^1$

$$\begin{array}{rcl} \gamma : \mathbb{P}^1 & \dashrightarrow & \overline{\mathcal{H}}_g(\mu) \\ H' & \mapsto & [H' \cap S, x_1, \dots, x_n]. \end{array}$$

The map can be extended and, by Lemma 4.11, it cannot be trivial. This gives us Theorem 4.2 for $g \leqslant 9$ and $g = 11$, when the length of μ is at least $g-1$.

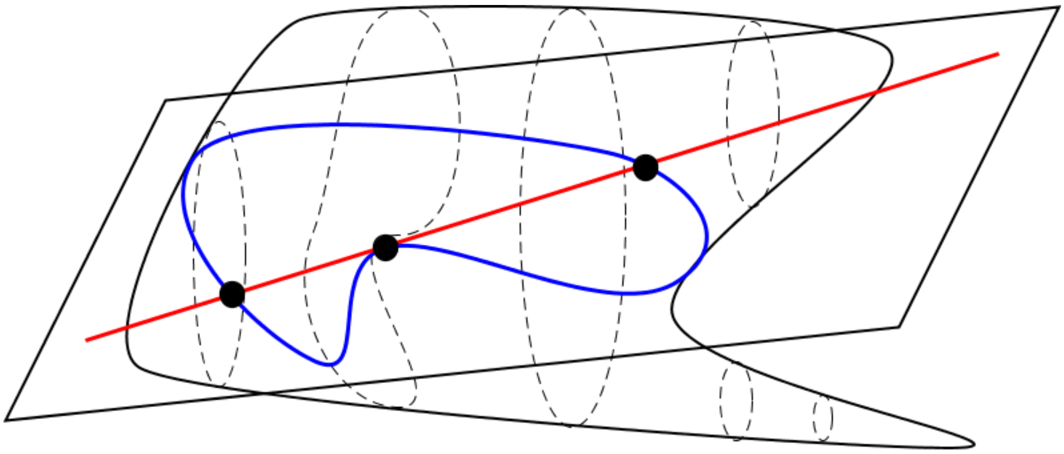

Figure 4.2: Pencil of canonical curves with a fixed canonical divisor.

Special Cases for Genus $g \leqslant 8$.

Let C be a smooth curve of genus $g \geqslant 2$. An *ample K3 extension* of C is a K3 surface S with at worst rational double points which contains C in the smooth locus as ample divisor. Let us recall a result of Ide.

Theorem 4.12 ([Ide08]). *All smooth curves of genus $2 \leqslant g \leqslant 8$ have ample* K3 *extensions. Moreover, they have smooth ample K3 extensions except in the following cases;*

- $g = 6, 7, 8$ *and* $K_C = 2D$ *where* D *is a* g^2_{g-1}*, or*
- $g = 8$ *and* $K_C = A + 2B$ *where* A *is a* g^1_4 *and* B *is a* g^1_5.

Every complex surface S with at worst ADE singularities admits a crepant resolution (see [Rei87]). In our case S is a K3 surface with at worst rational double points, so there is a unique resolution

$$\pi : \tilde{S} \to S.$$

The resolution is crepant meaning $\pi^* K_S = K_{\tilde{S}} = 0$ and $h^1(\mathcal{O}_S)$ is a birational invariant for surfaces with mild enough singularities. The smooth surface $\tilde{S}$ is again a K3 and if $\tilde{C}$ is the proper transform of C, as divisor $\tilde{C}$ might cease to be ample (it can have trivial intersection with (-2)-curves) but the self intersection is positive; it is still big and nef. We can rephrase Ide's theorem as follow.

Theorem 4.13. *Every smooth curve* C *of genus* $2 \leqslant g \leqslant 8$ *can be embedded in a smooth* K3 *surface* S *with* $C \subset S$ *big and nef.*

Let $[C, x_1, \dots, x_n] \in \mathcal{H}_g(\mu)$ be a general point on the stratum with $3 \leqslant g \leqslant 8$, $\mathcal{H}_g(\mu)$ connected and S be a big and nef K3 extension of C. Then the map $\phi_C : S \dashrightarrow \hat{S} \subset \mathbb{P}^g$ restricted to C is the canonical map

$$\phi_C\,|_C \colon C \to \mathbb{P}^{g-1}.$$

The point $[C, x_1, \dots, x_n]$ is general and we are under the assumption that $\mathcal{H}_g(\mu)$ is connected. Therefore, C is not hyperelliptic and ϕ_{K_C} is an embedding. We can repeat the same construction as for the general case. Since the general hyperplane of $\mathbb{P}^g$ in the pencil of hyperplanes through Λ_μ as before intersects $\hat{S}$ in a smooth curve, the pull back is smooth (it does not contains -2-curves).

This gives us uniruledness for every irreducible stratum $\overline{\mathcal{H}}_g(\mu)$ in the range $3 \leqslant g \leqslant 8$.

Genus $g = 9$.

We have already proven in §4.3.1 that $\mathcal{H}_9(\mu)$ is uniruled for $l(\mu) \geqslant 8$. We can improve the lower bound by one. For small length partitions the forgetful map $\pi : \mathcal{H}_g(\mu) \to \mathcal{M}_g$ is no longer dominant and, in order to carry out the argument above, one has to prove that the image of Mukai's map

$$\begin{array}{ccc} \mathcal{V}_{g,0} & \to & \mathcal{M}_g \\ (S, H, C) & \mapsto & [C] \end{array}$$

intersects $\pi(\mathcal{H}_g(\mu))$ on a non-empty open in $\pi(\mathcal{H}_g(\mu))$. A smooth complex curve of genus 9 can be realized as an hyperplane section of a K3 if the curve is not pentagonal (has no g^1_5), cf. [Muk10, Thm. A]. In particular the image of $\mathcal{V}_{9,0} \to \mathcal{M}_9$ contains the complement of the Brill-Noether divisor D^1_5 consisting of pentagonal curves. Recall that $\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_g\right)$ is generated by λ and the boundary divisors $\delta_{irr}, \delta_1, \dots, \delta_{\lfloor g/2 \rfloor}$ and the slope of a divisor $D = a\lambda - b_{irr}\delta_{irr} - \sum_{1 \leqslant i \leqslant \lfloor g/2 \rfloor} b_i \delta_i$ not containing any boundary components is defined to be

$$s(D) := \frac{a}{\min\{b_{irr}, b_1, \dots, b_{\lfloor g/2 \rfloor}\}}.$$

One can check that if D and D' are two divisors on $\mathcal{M}_g$, with $\overline{D}, \overline{D}'$ their closures inside $\overline{\mathcal{M}}_g$, then $\overline{D} \leqslant \overline{D}'$ implies that $s(\overline{D}) \leqslant s(\overline{D}')$. Mullane [Mul17, §5] computed the class of the closure of the image $\mathcal{H}_g(\mu) \to \mathcal{M}_g$ when $l(\mu) = g - 2$. In genus 9 the slope is strictly bigger than eight, in particular, bigger than the slope of the pentagonal locus (cf. [HM90])

$$s(\overline{D}^1_5) = 6 + \frac{12}{10}.$$

In any case, when $\overline{\mathcal{H}}_9(\mu)$ is irreducible and $l(\mu) = g - 2$, the classes

$$\pi_*\left[\overline{\mathcal{H}}_9(\mu)\right] \quad \text{and} \quad \overline{D}^1_5$$

are not proportional, moreover since the slope of $\pi_* \left[\overline{\mathcal{H}}_9(\mu)\right]$ is strictly bigger than $s(D_5^1)$, the image of $\overline{\mathcal{H}}_9(\mu)$ cannot be contained in the pentagonal locus. We can conclude that for a general point $[C, x_1, \dots, x_7] \in \mathcal{H}_9(\mu)$, with partition length $l(\mu) = 7$, the curve C can be embedded in a K3 surfaces and the argument above can still be carried out. This finishes the proof of Theorem 4.2 for $g = 9$.

Remark 4.14. By specialization, to have uniruledness for every holomorphic partition in genus nine, it would be enough to show that there is a Brill-Noether general curve in the non-hyperelliptic component of $\mathcal{H}_g(2g-2)$, but to construct a Brill-Noether general curve C admitting a subcanonical point is not a trivial task.

4.3.2 Uniruledness for quadratic differentials

Let S be the blow-up of $\mathbb{P}^2$ along $0 \leqslant r \leqslant 8$ many point in general position. The surface S is a del Pezzo surface and the class $-2K_S$ is ample. There is a moduli space for such surfaces $\mathscr{P}_r$ realized as the quotient of an open subset $\mathcal{U}$ of $(\mathbb{P}^2)^r$ by the group $PGL(3)$. The moduli space has dimension $\min\{2r-8, 0\}$ and over it sits a natural space

$$\mathscr{B}_r = \{(S, C) \mid S \in \mathscr{P}_r \text{ and } C \in |-2K_S| \text{ smooth and irreducible}\},$$

whose fibers over each del Pezzo surface $S \in \mathscr{P}_r$ are open subsets of the projective space $|-2K_S|$. Since $\chi(\mathcal{O}_S) = 1$ and, by Riemann-Roch and Kodaira vanishing,

$$\dim H^0\left(S, \mathcal{O}_S\left(-2K_S\right)\right) = \chi(\mathcal{O}_S) + 3K_S^2 = 28 - 3r.$$

The fiber dimension of the map $\mathscr{B}_r \to \mathscr{P}_r$ is $27 - 3r$ and the dimension of $\mathscr{B}_r$ is $19 - r$. On the other hand if $C \in |-2K_S|$ is a smooth irreducible curve on S, the genus of C satisfies

$$2g - 2 = C^2 + K_S \cdot C = 2K_S = 18 - 2r$$

and there is a natural map

$$\begin{array}{rcl} \psi_r : \mathscr{B}_r & \to & \mathcal{M}_{10-r} \\ (S, C) & \mapsto & [C]. \end{array}$$

Proposition 4.15. *When $4 \leqslant r \leqslant 7$ the map ψ_r is dominant.*

Proof. Let $[C] \in \mathcal{M}_g$ be a general smooth curve with $3 \leqslant g \leqslant 6$. The Brill-Noether number $\rho(g, 2, 6) \geqslant 0$ and the general curve $[C] \in \mathcal{M}_g$ has a plane nodal model $\Gamma \subset \mathbb{P}^2$ of degree 6 with $10 - g$ nodes. Take $r = 10 - g$, $S = \mathrm{Bl}_r\mathbb{P}^2$ the blow-up of $\mathbb{P}^2$ along the nodes, $E_1, \dots, E_r$ the exceptional divisors and L the proper transform of the line. Then the proper transform of Γ is smooth and lies in the linear system $|-2K_S| = |6L - 2E_1 - \dots - 2E_r|$. □

Lemma 4.16. *Let $\nu = (n_1, \dots, n_{m+g})$ be a partition of $4g-4$, of length $m+g$ and at least one odd entry. The forgetful map*

$$\begin{array}{rcl} \mathcal{Q}(\nu) & \to & \mathcal{M}_{g,m} \\ [C, p_1, \dots, p_{m+g}] & \mapsto & [C, p_1, \dots, p_m]\,. \end{array}$$

is dominant.

Proof. The diagram

$$\begin{tikzcd} \mathcal{Q}(\nu) \arrow[r, "i"] \arrow[d, "\pi"] & \mathcal{M}_{g,n+g} \arrow[d, "\sigma_\nu"] \\ \mathcal{M}_{g,n} \arrow[r, "c"] & \mathscr{J}ac_n^{2\cdot(2g-2)} \end{tikzcd}$$

is cartesian, where $\mathscr{J}ac_n^{4g-4}$ is the universal jacobian of degree $4g-4$ over $\mathcal{M}_{g,n}$, the map c is the 2-canonical section and σ_ν is the global Abel-Jacobi map given by

$$\sigma_\nu : [C, p_1, \dots, p_{m+g}] \mapsto \left[C, p_1, \dots, p_m, \mathcal{O}_C\left(\sum_{i=1}^{m+g} n_i p_i\right)\right].$$

For a smooth curve with marked points $[C, p] \in \mathcal{M}_{g,m}$, we fix

$$L_{[C,p]} = \mathcal{O}_C\left(\sum_{i=1}^{m} n_i p_i\right).$$

For dimension reasons the locus of curves $[C, p] \in \mathcal{M}_{g,m}$ such that $\omega^{\otimes 2} - L_{[C,p]}$ is supported at less than g points is of codimension at least one. Then, if $[C, p]$ is general in $\mathcal{M}_{g,m}$, the image of the restriction σ_ν to the fiber of the map

$$\mathscr{J}ac_n^{4g-4} \to \mathcal{M}_{g,m}$$

over a point $[C, p]$, that is,

$$\begin{array}{rcl} \sigma_\nu : C^{\times g} \setminus \triangle & \to & \mathrm{Pic}^{4g-4}(C) \\ (p_{m+1}, \dots, p_{m+g}) & \mapsto & L_{[C,p]} + \mathcal{O}_C\left(n_{m+1}p_{m+1} + \ldots + n_{m+g}p_{m+g}\right), \end{array}$$

contains $\omega^{\otimes 2}$. Therefore, the Abel-Jacobi map σ_ν dominates the image of the 2-canonical section c. □

Proof of Theorem 4.3. Let $g \leqslant 6$ and ν a partition of $4g-4$ with at least one non-even entry and length $l(\nu) \geqslant g$. Let $[C] \in \mathcal{Q}(\nu)$ be a general curve on the strata, then C is general in moduli so we can assume lies on the image of ψ_r with $r = 10 - g$. As before, the linear system $|-2K_S|$ embeds S in $\mathbb{P}^{3g-3}$ and realizes C as an hyperplane section

$$C = S \cap H \subset \mathbb{P}^{3g-3}.$$

The restriction of $-2K_S$ to C is $2K_C$ thus the map $S \hookrightarrow \mathbb{P}^{3g-3}$ restricted to C is the 2-canonical embedding and since $\sum n_i x_i \in \mathrm{Div}(C)$ is a quadratic differential, there must be a codimension 2 hyperplane $\Lambda_\mu \subset \mathbb{P}^{3g-3}$ with

$$\Lambda_\mu \cdot S = \sum n_i x_i.$$

Again let P be the pencil of hyperplanes $H \in (\mathbb{P}^{3g-3})^\vee$ containing Λ_μ. The points $x_1, \ldots, x_n$ lie on the base locus of this pencil and there is a rational map

$$\begin{array}{ccc} P & \dashrightarrow & \mathcal{Q}(\nu) \\ H' & \mapsto & [S \cap H', x_1, \ldots, x_n]\,. \end{array}$$

Is left to prove that the map is non-trivial. Let $\pi : \mathscr{U} \to \mathbb{P}^1$ be the family of curves induced by the pencil P, Z the base locus of P and $\varepsilon : \mathscr{U} \to S \to \mathbb{P}^2$ the composition of the blow-up of S at Z with the blow up map $\varepsilon : S \to \mathbb{P}^2$. If every fiber of π is smooth, then the topological Euler characteristic of $\mathscr{U}$ is

$$\chi(\mathscr{U}, \mathbb{Z}) = \chi(\mathbb{P}^1, \mathbb{Z}) \cdot \chi(\pi^{-1}(\text{point}), \mathbb{Z}) = 2 \cdot (2 - 2g).$$

But $\mathscr{U}$ is the composition of the blow up of $\mathbb{P}^2$ at r points together with the blow up at $|Z|$ points on S. Thus,

$$\chi(\mathscr{U}, \mathbb{Z}) = 3 + r + |Z| \geqslant 3$$

which is a contradiction. Thus, $\pi : \mathscr{U} \to \mathbb{P}^1$ must have singular fibers. This proves non-isotriviality. Fibers of π might still be curves isomorphic to C with a rational tail attached in which case the moduli map induced by the pencil is still trivial. But this cannot happen since if we see the pencil as a $\mathbb{P}^1$-family of r-nodal sextics in $\mathbb{P}^2$, if $C \sim R + C'$ where R is an irreducible rational tail, then R must be a line or a conic on $\mathbb{P}^2$ in which case the residual curve C' drops in genus. $\square$

Finally, recall that for genus $g = 4$ and holomorphic partition $\nu = (3, 3, 3, 3)$, the moduli space of quadratic differentials breaks into three connected components

$$\mathcal{Q}(\nu) = \mathcal{Q}^{hyp} \cup \mathcal{Q}^{irr} \cup \mathcal{Q}^{reg},$$

where

$$[C, x_1, \ldots, x_4] \in \mathcal{Q}^{irr} \quad \text{if and only if} \quad h^0\left(C, \mathcal{O}_C\left(\sum_1^4 x_i\right)\right) = 2$$

and

$$[C, x_1, \ldots, x_4] \in \mathcal{Q}^{reg} \quad \text{if and only if} \quad h^0\left(C, \mathcal{O}_C\left(\sum_1^4 x_i\right)\right) = 1.$$

Proof of Theorem 4.6. One can see that, the forgetful map $\mathcal{Q}^{reg} \to \mathcal{M}_4$ is dominant, and the same argument applies. As for $\mathcal{Q}^{irr}$, the forgetful map

$$\mathcal{Q}^{irr} \to \mathcal{M}_4$$

dominates a divisor and fibers of the induced map on the Σ_4-quotient correspond to pencils of the form $|x_1 + \ldots + x_4|$. $\square$

4.4 Uniruledness for $g = 10$

We define the open set $\mathcal{U}_{10} \subset \mathcal{H}_{10}(\mu)$ by the condition

$$[C, x_1, \ldots, x_n] \in \mathcal{U}_{10}$$

if and only if there exist a polarized K3 surface $[S, H] \in \mathcal{F}_{11}$ and a non trivial map $f : C \to S$, such that $f_*[C] \in |H|$ and f is the normalization map of the irreducible nodal curve $f(C)$ having a single node at $f(x_1) = f(x_2)$.

Proposition 4.17. *Every component of $\overline{\mathcal{U}}_{10} \subset \overline{\mathcal{H}}_{10}(\mu)$ is uniruled.*

Proof. Let $[C, x_1, \ldots, x_n]$ be a general point on $\mathcal{U}_{10}$ and

$$\epsilon : \tilde{S} \to S$$

the blow up of S at the node $f(x_1) = f(x_2)$. Then, the curve C can be embedded in $\tilde{S}$. Moreover, $C \in |\epsilon^* H - 2E|$, where E is the exceptional divisor of ϵ. By adjunction

$$\mathcal{O}_C(C) \cong K_C(-x_1 - x_2) \cong \mathcal{O}_C\left((m_1 - 1)x_2 + (m_2 - 1)x_2 + \sum_{i \geqslant 3} m_i x_i\right).$$

Let us assume that $l(\mu) \geqslant 2$. Then, the divisor

$$D = (m_1 - 1)x_2 + (m_2 - 1)x_2 + \sum_{i \geqslant 3} m_i x_i$$

is effective on C. The following sequence is exact

$$0 \to \mathcal{O}_{\tilde{S}} \to \mathcal{I}_{D/\tilde{S}}(C) \to \mathcal{O}_C \to 0,$$

where the middle term is the ideal sheaf of the closed subscheme $D \subset \tilde{S}$ twisted by C. Since $\tilde{S}$ is simply connected (recall that $h^{0,1}$ is a birational invariant of smooth surfaces)

$$\mathbb{P}H^0\left(\tilde{S}, \mathcal{I}_{D/\tilde{S}}(C)\right) \cong \mathbb{P}^1.$$

There is a rational map

$$\mathbb{P}^1 \dashrightarrow \mathcal{U},$$

sending the generic element $C' \in |\varepsilon^* H - 2E|$ passing through $\mathrm{Supp}(D)$ to

$$C' \in \mathbb{P}^1 \mapsto [C', x_1, x_2, \dots, x_n].$$

The same argument as in Lemma 4.11 applies to prove non-isotriviallity. We might assume $m_1 \geqslant m_2 \geqslant \dots \geqslant m_n$. Notice that the argument fails when the set $\{x_1, x_2\}$ is not contained in the support of D. If $m_1 > m_2 = 1$, we still can keep track of the points since $x_1 \in \mathrm{Supp}(D)$ and for C'' general, $C'' \cap E = x_1 + q$. We impose $x_2 = q$ and the argument still holds. □

When $m_2 = m_1 = 1$, then $\mu = (1, \dots, 1)$ with $l(\mu) = 18$ and our map is well defined in the quotient

$$\begin{array}{ccc} \mathbb{P}^1 & \dashrightarrow & \mathcal{U}_{10}/\mathbb{Z}_2 \\ C'' & \mapsto & [C'', y_1 + y_2, x_3, \dots, x_n] \end{array}$$

where $y_1 + y_2 = C'' \cap E$ and we have uniruledness for the quotient

$$\mathcal{U}_{10} \to \mathcal{U}_{10}/\mathbb{Z}_2.$$

Recall the second chapter; $\mathcal{V}_{11,1}$ is the moduli space of polarized K3's of genus 11 with a 1-nodal hyperplane section. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{V}_{11,1} \times \mathcal{H}_{10}(\mu) & \xrightarrow{p_2} & \mathcal{H}_{10}(\mu) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ \mathcal{V}_{11,1} & \xrightarrow{c} & \mathcal{M}_{10,[2]}. \end{array}$$

Notice that the image of p_2 is exactly $\mathcal{U}_{10}$ and, in the range $l(\mu) \geqslant g + 1$, the map π is dominant. But we already proved that c is dominant (Theorem. 2.4) and as corollary we have:

Corollary 4.18 (of Theorem 2.4). *For every holomorphic partition μ of length $18 > l(\mu) \geqslant 11$ the inclusion defined above $\mathcal{U}_{10} \hookrightarrow \mathcal{H}_{10}(\mu)$ is dominant.*

Corollary 4.19 (of Theorem 2.4). *For every partition of length $l(\mu) \geqslant 11$, the set $\mathcal{U}_{10} \subset \overline{\mathcal{H}}_{10}(\mu)$ is non-empty and contains an open dense subset. In particular $\overline{\mathcal{H}}_{10}(\mu)$ is uniruled for $11 \leqslant l(\mu) < 18$.*

4.5 Miscellanea

The next task in the study of the birational geometry of the strata is to determine when $\overline{\mathcal{H}}^k_g(\mu)$ is of general type or, at least, to establish a range for k, g and μ, where $\mathcal{H}_g^k(\mu)$ has no rationality properties. The possible strategies to attack such problem are essentially two. As we did for $\mathcal{H}_g^{hyp}(2g-2)$ and $\mathcal{F}_{11,n}$, one strategy consist on finding a birational model for $\overline{\mathcal{H}}_g^k(\mu)$ for which we know the Kodaira dimension. But, to do this for infinitely many g's seems unlikely. The second strategy would be to find a compactification of the strata $\widehat{\mathcal{H}}_g^k(\mu)$ with singularities mild enough so that pluricanonical forms on the smooth locus

$$\omega \in H^0\left(\widehat{\mathcal{H}}_g^k(\mu)^{reg}, nK_{\widehat{\mathcal{H}}_g^k(\mu)^{reg}}\right)$$

extend to $\widehat{\mathcal{H}}_g^k(\mu)$. After that, showing non-negative Kodaira dimension translate into proving that $K_{\widehat{\mathcal{H}}_g^k(\mu)}$ is $\mathbb{Q}$-effective and showing that the strata is of general type translate into being able to decompose the canonical class

$$K_{\widehat{\mathcal{H}}_g^k(\mu)} = A + E,$$

wehre A is $\mathbb{Q}$-ample and E is a $\mathbb{Q}$-effective divisor. In any case, having a good understanding of the canonical class of the strata is essential. The goal of this section is to give some partial results in this direction.

4.5.1 Finite maps to $\mathcal{M}_{g,n}$

Let $n, s \geqslant 0$, $k \geqslant 1$ and $g \geqslant 3$ be positive integers and

$$\mu = (m_1, \dots, m_n, m_{n+1}, \dots, m_{n+s})$$

a primitive partition of $k \cdot (2g-2)$ with length $l(\mu) = n + s$. Let

$$\pi : \mathcal{H}_g^k(\mu) \to \mathcal{M}_{g,n}$$

be the restriction of the map $\mathcal{M}_{g,n+s} \to \mathcal{M}_{g,n}$ that forgets the last s marked points. The goal of this section is to compute the degree of π when the map is generically finite. As already treated in §1.3.1, by dimension reasons, if $k = 1$ and every entry of μ is positive, the source and target of π have the same dimension when $s = g - 1$. In all other cases source and target have the same dimension for $s = g$.

Proposition 4.20. *The forgetful map*

$$\pi : \mathcal{H}_g^k(\mu) \to \mathcal{M}_{g,n}$$

is generically finite if

- $k \geqslant 2$ *and* $l(\mu) = n + g$, *or*
- $k = 1$ *and* μ *is meromorphic of length* $l(\mu) = n + g$, *or*

- $k = 1$ *and* μ *is holomorphic of length* $l(\mu) = n + (g-1)$, *with the only exception of the spin partition* $(2, \dots, 2)$.

Proof. The first two cases follow from the diagram (4.5) and the generic finiteness of the Abel Jacobi map

$$\begin{array}{rcl} AJ_\mu : C^g & \to & \mathrm{Jac}(C) \\ (p_{n+1}, \dots, p_{n+g}) & \mapsto & \sum_{i=n+1}^{n+g} m_i(p_i - p_0). \end{array}$$

We will compute the degree of AJ_μ in the coming Lemma 4.21. For the case $k = 1$ and μ holomorphic we do it by induction on n. The base case $n = 0$ is proved in [Gen15, Thm. 5.7], where Gendron used a chain of elliptic curves $E_1 \cup \dots \cup E_g$, with first and last member of the chain (E_1, N_1) and (E_g, N_g) having one marked point on them and the intermediate ones $(E_i, N_{i,1}, N_{i,2})$ having two marked points, such that E_i and E_{i+1} are glued along $N_{i,2} \sim N_{i+1,1}$ and for each E_i the line bundle associated to the divisor $N_{i,1} - N_{i,2}$ is torsion of order $2i$. One can check that, for such a curve there are only finitely many preimages. Assume that we have generic finiteness for $l(\mu) = n - 1 + (g-1)$. We consider the partial compactification, where we allow the points to come together. This is the same as allowing rational tails, under the identification,

$$\mathcal{M}_{g,n}^{rt} \cong \mathcal{C}_g^{\times n},$$

where $\mathcal{C}_g \to \mathcal{M}_g$ is the universal curve and the fiber product is taken over $\mathcal{M}_g$. The partially compactified

$$\mathcal{H}_g(\mu)^{rt} \subset \mathcal{C}_g^{\times n+(g-1)}$$

can be interpreted as $\mathcal{H}_g(\mu)$, where we allow points to come together. By induction hypothesis, the forgetful map

$$\pi : \mathcal{H}_g(\mu)^{rt} \to \mathcal{C}_g^{\times n},$$

restricted to a diagonal $D_{i=j}$, where $i, j \in \{1, \dots, n\}$, is generically finite. Since the pull back of $D_{i=j}$ under the map $\mathcal{C}_g^{\times n+(g-1)} \to \mathcal{C}_g^{\times n}$ is again the diagonal $D_{i=j}$ and, source and target have the same dimension, the full map π must be generically finite. $\square$

In what follows we compute the degree of π when generic finiteness holds. We start studying the case $l(\mu) = n + g$ with

$$\dim \mathcal{H}_g^k(\mu) = \dim \mathcal{M}_{g,n} = 3g - 3 + n.$$

We fix a general genus g curve C and a point $p_0 \in C$ on the curve. Consider the Abel-Jacobi map associated to the partition μ

$$\begin{array}{rcl} AJ_\mu : C^g & \to & \mathrm{Jac}(C) \\ (p_{n+1}, \dots, p_{n+g}) & \mapsto & \sum_{i=n+1}^{n+g} m_i(p_i - p_0). \end{array}$$

Lemma 4.21. *The map* AJ_μ *is generically finite of degree*

$$\deg(AJ_\mu) = g! \prod_{i=n+1}^{n+g} m_i^2.$$

Proof. The map AJ_μ seats in the following diagram

$$\begin{tikzcd}
\mathrm{Jac}(C)^{\times g} \arrow[r, "\prod \mathrm{mult}_{m_i}"] & \mathrm{Jac}(C)^{\times g} \arrow[r, "m"] & \mathrm{Jac}(C) \\
& C^g \arrow[ul, "\prod e"] \arrow[u, "\prod F_i"] \arrow[ur, "AJ_\mu"'] &
\end{tikzcd} \tag{4.3}$$

where

$$e(p) = p - p_0, \;\; F_i(p) = m_i(p - p_0) \;\text{ and }\; \mathrm{mult}_n(D) = nD.$$

Recall that

$$e_* [C] = \frac{\theta^{g-1}}{(g-1)!} = w_1$$

where θ is the theta divisor on $\mathrm{Jac}(C)$. If $L = -L$, as consequence of the Theorem of the Cube we have

$$\mathrm{mult}_n^* L = n^2 L.$$

These two facts, together with

$$m_* \left(\prod w_1 \right) = \theta^g = g!,$$

give us our result. ☐

Corollary 4.22. *Let* $k \geqslant 2$ *and* $l(\mu) = n + g$*, or* $k = 1$ *and* μ *is meromorphic and of length* $l(\mu) = n + g$*. We assume* $\mathcal{H}_g^k(\mu)$ *irreducible. Then the degree of the forgetful map*

$$\pi : \mathcal{H}_g^k(\mu)^{rt} \to \mathcal{M}_{g,n}^{rt}$$

is given by

$$\deg(\pi) = g! \prod_{i=n+1}^{n+g} m_i^2$$

Similarly, one can compute the degree of π when $k = 1$ and μ is holomorphic. Recall that, for a general curve C of genus g, the virtual number of sections of a g_d^r with ordered zeros of vanishing $k_1, \dots, k_{d-r}$ and $\sum k_i = d$ is given by De Jonquières' formula (cf. [ACGH85, VIII §5] or [Coo31, p. 288])

$$\frac{g!}{(g-d+r-1)!} \prod_{i=1}^{d-r} k_i \left(\sum_{j=0}^{d-r-1} \left(\frac{(-1)^j}{g-d+r+j} \sum_{|I|=j} \prod_{i \notin I} k_i \right) + \frac{(-1)^{d-r}}{g} \right).$$

One deduces that, for a general point $[C, x_1, \ldots, x_n] \in \mathcal{M}_{g,n}$, the degree of π is given by the number of sections of

$$\omega_C\left(-\sum_{i=1}^{n} m_i x_i\right)$$

with vanishing $m_{n+1}, \ldots, m_{n+(g-1)}$. This is,

$$\deg(\pi) = g! \prod_{i=n+1}^{g-1} m_i \left(\sum_{j=0}^{g-2}\left(\frac{(-1)^j}{j+1}\sum_{|I|=j}\prod_{i\notin I} m_{n+i}\right) + \frac{(-1)^{g-1}}{g}\right),$$

if $m_1 + \ldots + m_n \leqslant g-2$, and $\deg(\pi) = (g-1)!$, if $m_1 + \ldots + m_n = g-1$.

4.5.2 The canonical class of the strata for partitions of length $l(\mu) \geqslant g$

As explained above, when μ is holomorphic, $k = 1$, and the length of the partition is

$$l(\mu) = n + g - 1,$$

the forgetful map

$$\mathcal{H}_g^k(\mu) \to \mathcal{M}_{g,n} \tag{4.4}$$

is generically finite. The same holds when μ is primitive, meromorphic or $k \geqslant 2$, and of length $l(\mu) = n + g$. The aim of this section is to compute the ramification divisor of the map (4.4).

Definition 4.23. We define the *tautological Picard group of the strata*

$$R^1(\mathcal{H}_g^k(\mu)) \subset \mathrm{Pic}(\mathcal{H}_g^k(\mu)) \otimes \mathbb{Q}$$

to be the restriction of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{g,l(\mu)})$ to $\mathcal{H}_g^k(\mu)$. For the general definition of $R^i(\mathcal{H}_g^k(\mu))$, see [Che17b].

Let μ be a primitive partition of $k \cdot (2g-2)$ with length $l(\mu) = n + s$. Recall that, our space of interest sits in the cartesian diagram

$$\begin{tikzcd} \mathcal{H}_g^k(\mu) \arrow[r, "i"] \arrow[d, "\pi"] & \mathcal{M}_{g,n+s} \arrow[d, "\sigma_\mu"] \\ \mathcal{M}_{g,n} \arrow[r, "c"] & \mathscr{J}ac_{g,n}^{k\cdot(2g-2)}, \end{tikzcd} \tag{4.5}$$

where $c : [C, p_1, \ldots, p_n] \mapsto [C, p_1, \ldots, p_n, \omega_C^{\otimes k}]$ is the k-canonical section and

$$\sigma_\mu : [C, p_1, \ldots, p_n, x_1, \ldots, x_s] \mapsto \left[C, p_1, \ldots, p_n, \mathcal{O}\left(\sum_1^n m_i p_i + \sum_1^s m_{n+i} x_i\right)\right]$$

is the universal Abel-Jacobi map associated to μ. Then,

$$\Omega^1_\pi = i^*\Omega^1_{\sigma_\mu}$$

and, when π is generically finite,

$$K_{\mathcal{H}^k_g(\mu)} = \pi^* K_{\mathcal{M}_g} + D_{g,\mu} \,|_{\mathcal{H}^k_g(\mu)} \tag{4.6}$$

where $D_{g,\mu}$ is the class corresponding to the degeneracy locus of the vector bundle map

$$d\sigma_\mu : T\mathcal{M}_{g,n+s} \to \sigma_\mu^* T\mathscr{J}ac^{k\cdot(2g-2)}_{g,n}$$

over $\mathcal{M}_{g,n+s}$. Notice that the map i is an embedding and, when $D_{g,\mu}$ has codimension two or higher, the restriction might still be a divisor in $\mathcal{H}^k_g(\mu)$. Moreover, by purity of the ramification locus [Stacks, Tag 0EA1, §52.21], the map (4.4) is always ramified over a divisor, regardless of the codimension of $D_{g,\mu}$ inside $\mathcal{M}_{g,n+s}$.

Proposition 4.24. *For $s \leqslant g$ the differential*

$$d\sigma_\mu : T\mathcal{M}_{g,n+s} \to \sigma_\mu^* T\mathscr{J}^{k\cdot(2g-2)}_n$$

is degenerate in codimension g-s+1. The locus where it degenerates is given by the pull back under the forgetful map $\mathcal{M}_{g,n+s} \to \mathcal{M}_{g,s}$ of the reduced divisor

$$D_s = \{[C, x_1, \dots, x_s] \in \mathcal{M}_{g,s} \mid h^0(C, x_1 + \dots + x_s) \geqslant 2\} \subset \mathcal{M}_{g,s}.$$

Remark 4.25. Notice that it does not depends on μ. When $s = g$, D_g is a divisor on $\mathcal{M}_{g,g}$ and, when $k = 1$ and μ has negative entries or when $k \geqslant 2$, the map (4.4) is ramified at the restriction to $\mathcal{H}^k_g(\mu)$ of the pull back of D_g to $\mathcal{M}_{g,n+g}$. The class of the closure of D_g inside $\overline{\mathcal{M}}_{g,g}$ was computed by Logan [Log03, Thm. 5.4] and it is given by

$$\overline{D}_g = -\lambda + \sum_{i=1}^{g} \psi_i - \sum_{\substack{i=0,\dots,\lfloor g/2 \rfloor \\ S\subset\{1,\dots,g\}}} \binom{||S|-i|+1}{2} \delta_{i:S} \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,g}). \tag{4.7}$$

As a corollary we have:

Corollary 4.26. *For $k = 1$ and μ meromorphic or $k \geqslant 2$ and μ primitive. When $l(\mu) \geqslant g$, the canonical class of the strata lies in $R^1(\mathcal{H}^k_g(\mu))$ and it is given by*

$$K_{\mathcal{H}^k_g(\mu)} = 12\lambda + \sum_{i=1}^{l(\mu)} \psi_i,$$

where λ and ψ_i are the restrictions to the strata of the standard tautological classes.

Proof. We assume $l(\mu) = n + g$. It is well known that

$$K_{\mathcal{M}_{g,n}} = 13\lambda + \sum_1^n \psi_i,$$

see [Log03]. Let $p_1 : \mathcal{M}_{g,n+g} \to \mathcal{M}_{g,n}$ be the map that forgets the last g marked points and $p_2 : \mathcal{M}_{g,n+g} \to \mathcal{M}_{g,g}$ the one that forgets the first n marked points. As in the diagram (4.5), π is the generically finite map in question and

$$i : \mathcal{H}_g^k(\mu) \to \mathcal{M}_{g,n+g}$$

the inclusion. Since $\pi = p_1 \circ i$, we have

$$K_{\mathcal{H}_g^k(\mu)} = i^* \left(p_1^*(K_{\mathcal{M}_{g,n}}) + p_2^* D_g\right).$$

By (4.7) and by the pull-back formulas for tautological classes (see [ACG11, Ch. XVII]) we have

$$K_{\mathcal{H}_g^k(\mu)} = i^* \left(12\lambda + \sum_{i=1}^{n+g} \psi_i\right).$$

$\square$

Remark 4.27. As we will see in the next section, the dependency of $K_{\mathcal{H}_g^k(\mu)}$ with the partition μ is hidden in the relations between λ and the ψ_i classes.

Proof of Proposition 4.24. We fix a general curve $[C] \in \mathcal{M}_g$, general point $x_0 \in C$ and a basis $\{\omega_1, \dots, \omega_g\}$ of $H^0(C, K_C)$. Let $AJ : C^{\times s} \to \mathrm{Jac}(C)$ be the Abel-Jacobi map associated the truncated partition $\mu_{>n} = (m_{n+1}, \dots, m_{n+s})$. In the analytic setting

$$AJ(x_1, \dots, x_s) = \left(\sum_{i=1}^s m_{n+i} \int_{p_0}^{x_i} \omega_1, \dots, \sum_{i=1}^s m_{n+i} \int_{p_0}^{x_i} \omega_g\right) \quad \mathrm{mod}\ H_1(C, \mathbb{Z}).$$

The differential of this map does not depends on p_0 and it is given by

$$dAJ\,|_{(x_1,\dots,x_s)} = \begin{pmatrix} \omega_1(x_1) & \dots & \omega_1(x_s) \\ \vdots & \ddots & \vdots \\ \omega_g(x_1) & \dots & \omega_g(x_s) \end{pmatrix}_{g\times s} \cdot \mathrm{diag}(m_{n+1}, \dots, m_{n+s}).$$

In local coordinates, if $\omega_i = f_i\, dz$, with z a centered coordinate around x_i, then $\omega_i(x_j) = f_i(0)$. Disregarding diagonals, dAJ cease to have maximal rank only when there is an holomorphic 1-form $\omega \in H^0(C, K_C)$ vanishing at every point $x_1, \dots, x_s$, that is, when $h^0(K_C - x_1 - \dots - x_s) \geqslant 1$. By Riemann-Roch, if

$$D_s = \{(x_1, \dots, x_s) \in C^s \setminus \triangle \mid \mathrm{rk}\left(dAJ_{(x_1,\dots,x_s)}\right) \leqslant s-1\}$$

is the locus where the differential degenerates, then

$$D_s = \{(x_1, \ldots, x_s) \in C^k \setminus \triangle \mid h^0(x_1 + \ldots + x_s) \geqslant 2\}.$$

The locus D_s might be empty, if not, at the generic point of D_s,

$$h^0(K_C - x_1 - \ldots - x_k) = 1$$

and the differential form vanishes with order one at the points x_i. Thus, the class of the degeneracy locus $D_s \in A^{g-s+1}(C^s \setminus \triangle)$ is reduced.

Now lets fix a pointed curve $[C, p_1, \ldots, p_n, x_1, \ldots, x_s] \in \mathcal{M}_{g,n+s}$ and consider the splitting of the tangent spaces

$$\begin{tikzcd}
0 \arrow[r] & \oplus T_{x_i} C \arrow[r] \arrow[d, "dAJ"] & T_{[C,p,x]}\mathcal{M}_{g,n+s} \arrow[r] \arrow[d, "d\sigma_\mu"] & T_{[C]}\mathcal{M}_{g,n} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & H^1(\mathcal{O}_C) \arrow[r] & T_{[C,p,\mathcal{O}(\sum_1^n m_i p_i + \sum_1^s m_{n+i} x_i)]}\mathscr{I}_n^{k\cdot(2g-2)} \arrow[r] & T_{[C]}\mathcal{M}_{g,n} \arrow[r] & 0.
\end{tikzcd}$$

The map $d\sigma_\mu$ fails to have maximal rank at $[C, p_1, \ldots, p_n, x_1, \ldots, x_s]$ if and only if the Abel-Jacobi map dAJ degenerates at $(x_1, \ldots, x_s)$, that is, if

$$h^0(x_1 + \ldots + x_s) \geqslant 2.$$

$\square$

Remark 4.28. In the holomorphic case with $k = 1$ and $l(\mu) = n + (g-1)$, the differential of the universal Abel-Jacobi map σ_μ degenerates in codimension two. Nevertheless, $\pi : \mathcal{H}_g(\mu) \to \mathcal{M}_{g,n}$ is either étale or ramified along a divisor.

Example 4.29. Let μ be the Weierstrass partition $\mu = (g, 1, \ldots, 1)$. Consider the locus of pointed curves $[C, x_1, \ldots, x_{g-1}] \in \mathcal{M}_{g,g-1}$ where C a general hyperelliptic curve of even genus g, x_1 is a Weierstrass point on C and $(x_2, x_3), \ldots, (x_{g-2}, x_{g-1})$ are pairwise distinct points related by the hyperelliptic involution. The dimension of such locus in $\mathcal{M}_{g,g-1}$ is the dimension of the moduli of hyperelliptic curves $2g-1$ plus the choice of $(g-2)/2$ points on it. When $g = 4$ this is codimension one in $\mathcal{H}_g(\mu)$ and the points move in a pencil, i.e., $h^0(x_1 + \ldots + x_{g-1}) \geqslant 2$. Therefore, the map π is ramified over this locus.

The example above shows that when $k = 1$ and $l(\mu) = n + g - 1$, we cannot ensure the ramification divisor of

$$\mathcal{H}_g(\mu) \to \mathcal{M}_{g,n}$$

to lie in the tautological ring $R^1(\mathcal{H}_g(\mu))$. The shovel bends at this point because we don't have a description of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{H}_g(\mu))$, we don't know the rank, neither its generators.

4.5.3 Picard relations

The goal of this section is to establish relations among divisor classes in the strata that come from divisors on $\mathcal{M}_{g,n}$. This section follows the expositions in [Che17b] and [Sau17].

Proposition 4.30. *Let* $\mu = (m_1, \ldots, m_n)$ *be a partition of* $k \cdot (2g-2)$. *For any* $i \in \{1, \ldots, n\}$, *the following relation holds in* $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{H}_g^k(\mu))$

$$12k^2\lambda + \sum_{i=1}^{n}(k^2 + m_i^2)\psi_i = k \cdot (4g-4)(k+m_i)\psi_i.$$

Sauvaget proved in [Sau17, §5 Thm. 6] that, in the open strata, ψ classes are proportional. More precisely, when $k = 1$ and μ is holomorphic, for all $i, j \in \{1, \ldots, n\}$,

$$(m_i + 1)\psi_i = (m_j + 1)\psi_j \text{ in } \mathrm{Pic}_{\mathbb{Q}}\left(\mathcal{H}_g(\mu)\right).$$

As a corollary of the proposition above we have:

Corollary 4.31. *Let* $\mu = (m_1, \ldots, m_n)$ *be a partition of* $k \cdot (2g-2)$ *where negative entries are allowed. Then, for every* $i, j = 1, \ldots, n$

$$(k + m_i)\psi_i = (k + m_j)\psi_j \quad \textit{in} \quad \mathrm{Pic}\left(\mathcal{H}_g^k(\mu)\right) \otimes \mathbb{Q}.$$

Corollary 4.32. *If no entry of* μ *equals* $-k$, *then* $R^1(\mathcal{H}_g^k(\mu))$ *is generated by a single (possibly trivial) element. If some entry of* μ *equals* $-k$, *then* $R^1(\mathcal{H}_g^k(\mu))$ *is generated by all* ψ_i *with* $m_i = -k$.

As corollary we have the following theorem:

Theorem 4.33. *Let* $\mu = (m_1, \ldots, m_{l(\mu)})$ *be a partition of length* $l(\mu) \geqslant g$. *We assume the partition* μ *to be meromorphic if* $k = 1$ *and primitive if* $k \geqslant 2$. *Then the canonical class of the open strata lies in* $R^1(\mathcal{H}_g^k(\mu))$ *and it is given by*

$$K_{\mathcal{H}_g^k(\mu)} = c_\mu \lambda,$$

where

$$c_\mu = 12 + \frac{12k^2\left(\sum \frac{1}{k+m_i}\right)}{k \cdot (4g-4) - \sum \frac{k^2+m_i^2}{k+m_i}} \in \mathbb{Q}$$

if $m_i \neq -k$ *for all* $i = 1, \ldots, l(\mu)$, *and*

$$c_\mu = 6$$

if some $m_i = -k$.

Proof. Using the relations of Corollary 4.31 and Proposition 4.30, and the expression for the canonical class in Corollary 4.26, one deduces the first case. For the second case, we assume that the partition has the following shape

$$\mu = (\underbrace{-k, \dots, -k}_{r}, m_{r+1}, \dots, m_{l(\mu)}).$$

From Corollary 4.31 we have that $\psi_i = 0$ for $i > r$ and, from Proposition 4.30, we have

$$12k^2\lambda + 2k^2 \sum_{i=1}^{r} \psi_i = 0.$$

Again, from Corollary 4.26, one deduces that $c_\mu = 12 - \frac{12k^2}{2k^2} = 6$. □

Proof of Proposition 4.30. Let $\pi : \mathscr{C} \to B$ be a family of smooth genus g curves over a smooth base and

$$s_1, \dots, s_n : B \to \mathscr{C}$$

pairwise disjoint sections such that

$$\omega_\pi^{\otimes k} \cong \mathcal{O}_{\mathscr{C}}\left(\sum_{i=1}^{n} m_i D_i\right) + \pi^*\mathscr{L} \tag{4.8}$$

where $D_i = s_{i*}[B]$ and $\mathscr{L}$ is a line bundle on B. Recall that $\kappa = \pi_*(c_1(\omega)^2)$ and, since the fibers of π are smooth,

$$\lambda = \frac{1}{12}\left(\kappa - \sum_{i=1}^{n} \psi_i\right),$$

where λ is the first Chern class of the Hodge bundle on B. Notice that $\pi_*(\pi^*\mathscr{L})$ is trivial and $\pi_*(\mathscr{L} \mid D_i) = \mathscr{L}$. Thus,

$$k^2\kappa = -\sum m_i^2\psi_i + k \cdot (4g-4)c_1(\mathscr{L}).$$

On the other hand, if we restrict (4.8) to D_i and then push down to B, we get

$$c_1\left(\pi_*(\omega^{\otimes k}\mid_{D_i})\right) = -m_i\psi_i + c_1(\mathscr{L}). \tag{4.9}$$

This gives us

$$c_1(\mathscr{L}) \cong (m_i + k)\psi_i.$$

□

An alternative construction that gives us the same set of relations goes as follows:

Let $\pi : \mathscr{C} \to \mathcal{M}_{g,n}$ be the universal curve over $\mathcal{M}_{g,n}$ and

$$s_1, \dots, s_n : \mathcal{M}_{g,n} \to \mathscr{C}$$

the sections corresponding to the marked points. Let $D_i = s_{i*}[\mathcal{M}_{g,n}]$ be the associated divisors. We fix a line bundle $\mathscr{L}$ on $\mathscr{C}$ such that, when restricted to the fibers of π, is base point free. There is a natural evaluation map of sheaves on $\mathcal{M}_{g,n}$

$$ev_i : \pi_*\mathscr{L} \to \pi_* \left(\mathscr{L}\mid_{D_i}\right).$$

Notice that $\pi_* \left(\mathscr{L}\mid_{D_i}\right) = \mathscr{E}_i$ is a line bundle over $\mathcal{M}_{g,n}$. Now consider the projective cone

$$p : \mathbb{P}\left(\pi_*\mathscr{L}\right) \to \mathcal{M}_{g,n}.$$

There is a natural section of $\mathcal{O}(1) \otimes p^*\mathscr{E}_i$ given by

$$\begin{aligned} s : \mathcal{O}(-1) &\to p^*\mathscr{E}_i \\ \alpha &\mapsto ev_i(\alpha). \end{aligned}$$

The class of $\{s = 0\}$ is the one of the subvariety

$$\mathbb{P}(\pi_*(\mathscr{L} \otimes \mathcal{O}(-D_i))) \subset \mathbb{P}(\pi_*\mathscr{L}).$$

Thus, if ξ is the first Chern class of $\mathcal{O}(1)$, then

$$[\mathbb{P}(\pi_*(\mathscr{L}(-D_i)))] = \xi + p^*c_1(\mathscr{E}_i) \text{ in } A^1(\mathbb{P}(\pi_*\mathscr{L})).$$

Replacing $\mathscr{L}$ by $\omega_\pi^{\otimes k}$ $(+\text{poles} - \text{zeros})$ we have an alternative proof of Corollary 4.31. Indeed, let

$$\mathscr{L} := \omega_\pi^{\otimes k}\left(-\sum_{i=1}^{n} m_i D_i\right).$$

The evaluation map goes to

$$\pi_*\mathscr{L}\mid_{D_i} = \mathscr{L}_i^{\otimes m_i + k},$$

where $\mathscr{L}_i$ is the line bundle whose fiber over $[C, x_1, \dots, x_n]$ is the cotangent line of C at x_i. The line bundle $\mathscr{L}$ has fiberwise degree zero, thus,

$$\pi_*\mathscr{L}(-D_i) = 0$$

and

$$0 = \xi + p^*\psi_i^{m_i+k}.$$

But notice that $\pi_*\mathscr{L}$ is supported on $\mathcal{H}_g^k(\mu)$, since

$$h^0\left(C, \omega_C^{\otimes k}\left(-\sum m_i x_i\right)\right) = 1$$

if and only if $[C, x_1, \dots, x_n]$ lies on the strata $\mathcal{H}_g^k(\mu)$. Moreover,

$$p : \mathbb{P}(\pi_* \mathscr{L}) \to \mathcal{M}_{g,n}$$

is a close embedding with image $\mathcal{H}_g^k(\mu)$. From this follows that

$$(m_i + k)\psi_i = (m_j + k)\psi_j = -p_* \xi \,|_{\mathcal{H}_g^k(\mu)}$$

in $\mathrm{Pic}_{\mathbb{Q}}\left(\mathcal{H}_g^k(\mu)\right)$. This construction, in the case $k = 1$, was used in [Sau17, §5] to extend this relations to $\overline{\mathcal{H}}_g(\mu)$.

CHAPTER 5

Appendix

5.1 Mukai models

The following table summarizes the list of projective varieties V_g and vector bundles $\mathscr{E}_g$ on V_g, such that the generic member of $\mathcal{F}_g$ is given by the zero locus of a global section of $\mathscr{E}_g$ on V_g. The table was taken from [PSY16].

g	V_g	$\mathscr{E}_g$	g	V_g	$\mathscr{E}_g$
2	$\mathbb{P}(1,1,1,2)$	$\mathcal{O}(6)$	9	$\mathrm{Gr}(3,6)$	$\mathcal{O}(1)^{\oplus 4} \oplus \bigwedge^2 \mathcal{Q}$
3	$\mathbb{P}^3$	$\mathcal{O}(4)$	10	$\mathrm{Gr}(2,7)$	$\mathcal{O}(1)^{\oplus 3} \oplus \bigwedge^4 \mathcal{Q}$
4	$\mathbb{P}^4$	$\mathcal{O}(2) \oplus \mathcal{O}(3)$	12	$\mathrm{Gr}(3,7)$	$\mathcal{O}(1) \oplus \left(\bigwedge^2 \mathcal{E}^\vee\right)^{\oplus 3}$
5	$\mathbb{P}^5$	$\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$	13	$\mathrm{Gr}(3,7)$	$\left(\bigwedge^2 \mathcal{E}^\vee\right)^{\oplus 2} \oplus \bigwedge^3 \mathcal{Q}$
6	$\mathrm{Gr}(2,5)$	$\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$	16	$\mathcal{T}(2,3;\mathbb{C}^4)$	$\mathcal{E}_3^{\oplus 2} \oplus \mathcal{E}_2^{\oplus 2}$
7	$\mathrm{OGr}^+(5,10)$	$\mathcal{S}_1^{\oplus 8}$	18	$\mathrm{OGr}(3,9)$	$\mathcal{S}_2^{\oplus 5}$
8	$\mathrm{Gr}(2,6)$	$\mathcal{O}(1)^{\oplus 6}$	20	$\mathrm{Gr}(4,9)$	$\left(\bigwedge^2 \mathcal{E}^\vee\right)^{\oplus 3}$

- $\mathbb{P}(1,1,1,2)$ stands for the wighted projective space with weights $(1,1,1,2)$ and $\mathcal{O}(-1)$ is the tautological line bundle. Observe that any double cover of $\mathbb{P}^2$ ramified over a sextic is given by the restriction of the usual projection to the vanishing of such section, $Z(s) \subset \mathbb{P}(1,1,1,2) \to \mathbb{P}^2$.

- For any Grassmannian $\mathrm{Gr}(r,n)$, $\mathcal{O}(1)$ is the pull back of the hyperplane class under the Plücker embedding, $\mathcal{E}$ is the tautological bundle, whose fiber at $[U] \in \mathrm{Gr}(r,n)$ is given by the vector space U and $\mathcal{Q}$ is the universal quotient bundle. They are related by the sequence

$$0 \to \mathcal{E} \to \mathcal{O} \otimes_{\mathbb{C}} \mathbb{C}^n \to \mathcal{Q} \to 0.$$

- Let Q be a non-degenerate, symmetric, bilinear form on a complex vector space V. The space $\mathrm{OGr}(r, V) \subset \mathrm{Gr}(r, V)$ parameterizes r-dimensional isotropic subspaces of (V, Q), that is, subspaces where Q vanishes. This variety can also be interpreted as the Fano variety of $r-1$ linear subspaces of the vanishing of Q in $\mathbb{P}(V)$. They are homogenous spaces for the group $\mathrm{Spin}(V)$. This means that

$$\mathrm{OGr}(r, V) = \mathrm{Spin}(V)/P,$$

where P is certain *parabolic subgroup*. If $2r = \dim V$, $\mathrm{OGr}(r, V)$ has two connected components and $\mathrm{OGr}^+(r, V)$ is one of them. The pull back of $\mathcal{O}(1)$ by the induced Plücker embedding $\mathrm{OGr}^+ \hookrightarrow \mathrm{Gr} \hookrightarrow \mathbb{P}(\bigwedge^r V)$, in the case $2r = \dim V$, is twice the generator of $\mathrm{Pic}(\mathrm{OGr}^+(r, V))$. The ample generator is called $\mathcal{S}_1$. See [Ott88] or [Kuz08, §6] for more on *spinor varieties*. The bundle $\mathcal{S}_2$ correspond to the rank two spin bundle of $\mathrm{OGr}(3, 9)$. We refer to [Ott88, Thm. 2.8] or [Kuz08, Cor. 6.5 and Prop. 6.6] for the properties that define higher rank spin bundles and their relations with $\mathcal{O}(1)$.

- The moduli space $\mathcal{T}(2, 3; \mathbb{C}^4)$ is called *EPS moduli space of twisted cubics in* $\mathbb{P}^3$. It is constructed by Ellingsrud, Piene and Strømme in [EPS87]. For V a four dimensional $\mathbb{C}$-vector space, the moduli $\mathcal{T}(2, 3; V)$ is constructed as the GIT quotient of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes V$, by the induced action of $GL(2) \times GL(3)$ on the first two factors. A point $t \in \mathcal{T}$ represents an equivalence class of a 2×3-matrix with entries in V. By construction, $\mathcal{T}$ comes with two tautological bundles $\mathcal{E}_2$ and $\mathcal{E}_3$ of ranks 2 and 3 respectively, together with a tautological map

$$\mathcal{E}_3 \otimes V^\vee \to \mathcal{E}_2$$

and an isomorphim $\det\mathcal{E}_3 = \det\mathcal{E}_2$. We refer to [EPS87] for details.

Bibliography

[Ale96] V. Alexeev. "Moduli spaces $M_{g,n}(W)$ for surfaces". *Higher-Dimensional Complex Varieties (Trento, 1994)* **147**.3 (1996), 1–22 (cit. on p. 8).

[Ale06] V. Alexeev. "Higher-dimensional analogues of stable curves". *International Congress of Mathematicians, Vol. II, Eur. Math. Soc.* (2006), 515–536 (cit. on p. 8).

[ACG11] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Vol. II*. Grundlehren der Mathematischen Wissenschaften 267. Springer-Verlag, New York, 2011 (cit. on pp. 4, 22, 97).

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*. Grundlehren der Mathematischen Wissenschaften 267. Springer-Verlag, New York, 1985 (cit. on pp. 14, 94).

[AM72] M. Artin and D. Mumford. "Some elementary examples of unirational varieties which are not rational". *Proc. London Math. Soc.* **25**.3 (1972), 75–95 (cit. on p. 24).

[AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. "Smooth Compactification of Locally Symmetric Varieties, Second Edition". *Cambridge University Press* (1975) (cit. on pp. 8, 64).

[BB66] W. L. Baily and A. Borel. "Compactification of arithmetic quotients of bounded symmetric domains". *Annals of Mathematics* **84**.2 (1966), 442–528 (cit. on p. 7).

[BCGGM16a] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. "Compactification of strata of abelian differentials, arXiv: 1604.08834" (2016) (cit. on p. 35).

[BCGGM16b] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. "Strata of k-differentials, arXiv: 1610.09238" (2016) (cit. on pp. 20, 21, 35).

[Bar17a] I. Barros. "Geometry of the moduli space of n-pointed K3 surfaces of genus 11, arXiv: 1705.05290v3" (2017) (cit. on p. 38).

[Bar17b] I. Barros. "Uniruledness of Strata of Holomorphic Differentials in Small Genus, arXiv: 1702.06716" (2017) (cit. on p. 38).

[BHPV04] W. Barth, K. Hulek, C. Peters, and A. Van de Ven. *Compact Complex Surfaces*. Vol. 4. A Series of Modern Surveys in Mathematics. Springer, 2004 (cit. on pp. 5–7).

[BE91] D. Bayer and D. Eisenbud. "Graph curves". *Advances in Mathematics* **86** (1991), 1–40 (cit. on p. 69).

[Bea04] A. Beauville. "Fano threefolds and K3 surfaces". *Proceedings of the Fano Conference* (2004), 175–184 (cit. on pp. 41, 50, 54).

[BLMM17] N. Bergeron, Z. Li, J. Millson, and C. Moeglin. "The Noether-Lefschetz conjecture and generalizations". *Inventiones mathematicae* **208**.2 (2017), 501–552 (cit. on p. 34).

[Bir09] C. Birkar. "The Iitaka conjecture $C_{n,m}$ in dimension six". *Compos. Math.* **145**.6 (2009), 1442–1446 (cit. on p. 29).

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. "Existence of minimal models for varieties of log general type". *Journal of the American Mathematical Society* **23**.2 (2010), 405–468 (cit. on p. 26).

[BDPP13] S. Boucksom, J.P. Demailly, M. Paun, and T. Peternell. "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension". *Journal of Algebraic Geometry* **22**.2 (2013), 201–248 (cit. on p. 26).

[BV05] A. Bruno and A. Verra. "$\mathcal{M}_{15}$ is rationally connected". *In: Projective varieties with unexpected properties* (2005), 51–65 (cit. on p. 30).

[CH98] L. Caporaso and J. Harris. "Counting plane curves of any genus". *Inventiones mathematicae* **131** (1998), 345–392 (cit. on p. 80).

[CHM97] L. Caporaso, J. Harris, and B. Mazur. "Uniformity of rational points". *J. Amer. Math. Soc.* **10** (1997), 1–35 (cit. on p. 73).

[CR84] M. C. Chang and Z. Ran. "Unirationality of the moduli spaces of curves of genus 11, 13 (and 12)". *Inventiones Mathematicae* **76**.1 (1984), 41–54 (cit. on pp. 30, 65).

[CR86] M. C. Chang and Z. Ran. "The Kodaira dimension of the moduli space of curves of genus 15". *J. Differential Geom.* **24** (1986), 205–220 (cit. on p. 30).

[CR91] M. C. Chang and Z. Ran. "On the slope and Kodaira dimension of $\mathcal{M}_g$ for small g". *J. Differential Geom.* **34** (1991), 267–274 (cit. on p. 30).

[CH11] A. Chen and C.D. Hacon. "Kodaira dimension of irregular varieties". *Inventiones Mathematicae* **186**.3 (2011), 481–500 (cit. on p. 29).

[Che10] D. Chen. "Covers of elliptic curves and the moduli space of stable curves". *J. Reine Angew. Math.* **649** (2010), 167–205 (cit. on p. 19).

[Che17a] D. Chen. "Affine geometry of strata of differentials, arXiv: 1706.01142" (2017) (cit. on p. 34).

[Che17b] D. Chen. "Tautological ring of strata of differentials, arXiv: 1708.00519" (2017) (cit. on pp. 35, 95, 99).

[Che17c] D. Chen. "Teichmüller dynamics in the eyes of an algebraic geometer". *Proc. Sympos. Pure Math.* **95** (2017), 171–197 (cit. on pp. 16, 19).

[CFM13] D. Chen, G. Farkas, and I Morrison. "Effective divisors on moduli spaces of curves and abelian varieties". *Clay Math. Proc.* **18** (2013), 131–169 (cit. on p. 19).

[CM14] D. Chen and M. Möller. "Quadratic differentials in low genus: exceptional and non-varying strata". *Ann. Sci. Éc. Norm. Supér.* **47**.2 (4 2014), 309–369 (cit. on pp. 78, 79).

[CD12] C. Ciliberto and T. Dedieu. "On universal Severi varieties of low genus K3 surfaces". *Mathematische Zeitschrift* **271**.3 (2012), 953–960 (cit. on pp. 40, 67–70).

[CLM93] C. Ciliberto, A. Lopez, and R. Miranda. "Projective degenerations of K3 Surfaces, Gaussian maps, and Fano threefolds". *Inventiones Mathematicae* **114** (1993), 641–667 (cit. on p. 41).

[CM90] C. Ciliberto and R. Miranda. "On the Gaussian map for canonical curves of low genus". *Duke Mathematical Journal* **61**.2 (1990), 417–443 (cit. on p. 69).

[CJ16] E. Clader and F. Janda. "Pixton's double ramification cycle relations". *Geometry and Topology* **22** (2016) (cit. on p. 36).

[CG72] H. Clemens and P. Griffiths. "The intermediate Jacobian of the cubic threefold". *Annals of Mathematics* **95**.2 (1972), 281–356 (cit. on p. 24).

[Coo31] J. L. Coolidge. *A treatise on algebraic plane curves*. Oxford University Press, 1931 (cit. on p. 94).

[Deb01] O. Debarre. *Higher-Dimensional Algebraic Geometry*. Universitext. Springer-Verlag, New York, 2001 (cit. on p. 26).

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 75–109 (cit. on p. 32).

[Dia84] S. Diaz. “Tangent spaces in moduli via deformations with applications to Weierstrass points”. *Duke Mathematical Journal* **51**.4 (1984), 905–922 (cit. on p. 22).

[EH87] D. Eisenbud and J. Harris. “The Kodaira dimension of the moduli space of curves of genus $g \geqslant 23$”. *Inventiones Mathematicae* **90**.2 (1987), 359–387 (cit. on p. 30).

[EPS87] G. Ellingsrud, R. Piene, and S. A. Strømme. “On the variety of nets of quadrics defining twisted cubic curves”. *In: Space Curves, Lecture Notes in Mathematics, F. Ghione, C. Peskine and E. Sernesi, editors* **1266** (1987), 84–96 (cit. on p. 104).

[EMM15] A. Eskin, M. Mirzakhani, and A. Mohammadi. “Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space”. *Annals of Mathematics* **182**.2 (2015), 673–721 (cit. on p. 18).

[Far00] G. Farkas. “The geometry of the moduli space of curves of genus 23”. *Math. Annalen* **318** (2000), 43–65 (cit. on p. 30).

[Far09] G. Farkas. “Koszul divisors on moduli spaces of curves”. *American Journal of Mathematics* **131** (2009), 819–867 (cit. on p. 31).

[Far10a] G. Farkas. “Aspects of the birational geometry of $\mathcal{M}_g$”. *Geometry of Riemann surfaces and their moduli spaces, Surveys in Differential Geometry* **14** (2010), 57–111 (cit. on p. 30).

[Far10b] G. Farkas. “The birational type of the moduli space of even spin curves”. *Advances in Math.* **223**.2 (2010), 433–443 (cit. on p. 19).

[FK16] G. Farkas and M. Kemeny. “The generic Green-Lazarsfeld secant conjecture”. *Inventiones Mathematicae* **203** (2016), 265–301 (cit. on p. 39).

[FK17] G. Farkas and M. Kemeny. “The Prym-Green conjecture for torsion line bundles of high order”. *Duke Mathematical Journal* **166** (2017), 1103–1124 (cit. on p. 39).

[FP15] G. Farkas and R. Pandharipande. “The moduli space of twisted canonical divisors, arXiv: 1508.07940.” *J. Institute Math. Jussieu (to appear)* (2015) (cit. on p. 35).

[FP05] G. Farkas and M. Popa. “Effective divisors on $\overline{\mathcal{M}}_g$, curves on K3 surfaces, and the slope conjecture”. *Journal of Algebraic Geometry* **14** (2005) (cit. on pp. 31, 43, 74).

[FV13] G. Farkas and A. Verra. “The classification of universal Jacobians over the moduli space of curves”. *Commentarii Math. Helvetici* **88** (2013), 587–611 (cit. on pp. 31, 65, 71, 72, 74).

[FV14] G. Farkas and A. Verra. "The geometry of the moduli space of odd spin curves". *Annals of Mathematics* **180**.3 (2014), 927–970 (cit. on pp. 10, 78, 79).

[FV18] G. Farkas and A. Verra. "The universal K3 surface of genus 14 via cubic fourfolds". *Journal de Mathématiques Pures et Appliquées* **111** (2018), 1–20 (cit. on pp. 33, 65).

[Fil16] S. Filip. "Splitting mixed Hodge structures over affine invariant manifolds". *Annals of Mathematics* **183**.2 (2016), 681–713 (cit. on p. 18).

[FKGS08] F. Flamini, A. L. Knutsen, Pacienza G., and E. Sernesi. "Nodal curves with general moduli on K3 surfaces". *Communications in Algebra* **36**.1 (2008), 3955–3971 (cit. on pp. 41, 43, 46, 47, 49, 50, 53–55, 78).

[Fuj13] O. Fujino. "On maximal Albanese dimensional varieties". *Proc. Japan Acad.* **86** (2013), 92–95 (cit. on p. 29).

[Gen15] Q. Gendron. "The Deligne-Mumford and the Incidence Variety Compactifications of the Strata of $\Omega\mathcal{M}_g$, arXiv: 1503.03338" (2015) (cit. on pp. 84, 93).

[GHS07] V. Gritsenko, K. Hulek, and G.K. Sankaran. "The Kodaira dimension of the moduli space of K3 surfaces". *Inventiones Mathematicae* **169** (2007), 519–567 (cit. on pp. 8, 64).

[Har84] J. Harris. "On the Kodaira dimension of the moduli space of curves II: The even-genus case". *Inventiones Mathematicae* **75** (1984), 437–466 (cit. on p. 30).

[HM90] J. Harris and I. Morrison. "Slopes of effective divisors on the moduli space of stable curves". *Inventiones Mathematicae* **99** (1990), 321–355 (cit. on p. 86).

[HM82] J. Harris and D. Mumford. "On the Kodaira dimension of the moduli space of curves". *Inventiones Mathematicae* **67**.1 (1982), 23–86. ISSN: 0020-9910 (cit. on pp. 30, 32).

[Has00] B. Hassett. "Special cubic fourfolds". *Compositio Mathematica* 120 (2000), 1–23 (cit. on p. 64).

[Hir64] H. Hironaka. "Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: I". *Annals of Mathematics* **79**.1 (1964), 109–203 (cit. on p. 27).

[Hir89] A. Hirschowitz. "Une conjecture pour la cohomologie des diviseur sur les surfaces rationnelles génériques". *J. Reine Angew. Math.* 397 (1989), 208–213 (cit. on p. 61).

[Huy16] D. Huybrechts. *Lectures on K3 Surfaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016 (cit. on p. 4).

[Ide08] M. Ide. "Every curve of genus not grater than eight lies on a K3 surface". *Nagoya Math. J.* **190** (2008), 183–197 (cit. on p. 85).

[Iit81] S. Iitaka. *Algebraic Geometry: An introduction to birational geometry of algebraic varieties*. Springer-Verlag, 1981 (cit. on p. 29).

[IM71] V. Iskovskikh and Y. Manin. "Three-dimensional quartics and counterexamples to the Lüroth problem". *Math. USSR.Sb.* **15** (1971), 141–166 (cit. on p. 24).

[Kad17] İ. Kadıköylü. "Maximal Rank Divisors on $\overline{\mathcal{M}}_{g,n}$, arXiv: 1705.04250" (2017) (cit. on p. 31).

[Kaw79] Y. Kawamata. "The Kodaira dimension of certain fibre spaces". *Proc. Japan Academy* 5 (1979), 406–408 (cit. on pp. 29, 65).

[Kaw81] Y. Kawamata. "Characterization of abelian varieties". *Compositio Mathematica* **43**.2 (1981), 253–276 (cit. on p. 29).

[Kle81] J. O. Kleppe. "The Hilbert flag scheme, its properties and its connection with the Hilbert scheme. Applications to curves in 3-space". *PhD thesis, University of Oslo* (1981) (cit. on p. 67).

[Knu01] A. L. Knutsen. "On k-th order embeddings of K3 surfaces and Enriques surfaces". *Manuscripta Mathematica* **104** (2001), 211–237 (cit. on p. 60).

[Kol87] J. Kollár. "Subadditivity of the Kodaira dimension: fibers of general type". *Alg. geom., Sendai, 1985. Adv. Stud. Math.* **10** (1987), 361–398 (cit. on p. 29).

[KS88] J. Kollár and N. Shepherd-Barron. "Threefolds and deformations of surface singularities". *Inventiones Mathematicae* **91**.2 (1988), 299–338 (cit. on p. 8).

[KSC04] J. Kollár, K. E. Smith, and A. Corti. *Rational and nearly rational varieties*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004 (cit. on p. 24).

[Kon93] S. Kondo. "On the Kodaira dimension of the moduli space of K3 surfaces". *Compositio Mathematica* **89**.3 (1993), 251–299 (cit. on p. 64).

[KZ96] M. Kontsevich and A. Zorich. "Lyapunov Exponents and Hodge Theory". *Adv. Ser. Math. Phys.* **24** (1996), 318–332 (cit. on p. 15).

[KZ03] M. Kontsevich and A. Zorich. "Connected components of the moduli spaces of Abelian differentials with prescribed singularities". *Inventiones Mathematicae* **153** (2003), 631–678 (cit. on pp. 13, 20).

[Kuz08] A. Kuznetsov. "Exceptional collections for Grassmannians of isotropic lines". *Proc. Lond. Math. Soc.* **97**.1 (3 2008), 155–182 (cit. on p. 104).

[Lan02] E. Lanneau. "Hyperelliptic components of the moduli spaces of quadratic differentials with prescribed singularities". *Commentarii Mathematici Helvetici* **79**.4 (2002) (cit. on p. 78).

[Laz16] R. Laza. "The KSBA compactification for the moduli space of degree two K3 pairs". *J. Eur. Math. Soc.* **18**.2 (2016), 225–279 (cit. on p. 8).

[LO16] R. Laza and K. O'Grady. "Birational geometry of the moduli space of quartic K3 surfaces, arXiv: 1607.01324" (2016) (cit. on p. 8).

[LO17] R. Laza and K. O'Grady. "GIT versus Baily-Borel compactification for quartic K3 surfaces, arXiv: 1612.07432" (2017) (cit. on p. 8).

[Laz86] R. Lazarsfeld. "BrillNoetherPetri without degenerations". *Journal of Differential Geometry* **23** (1986), 299–307 (cit. on p. 39).

[Laz04] R. Lazarsfeld. *Positivity in algebraic geometry I. Classical Setting: Line bundles and linear series*. Ergebnisse der Mathematik 48. Springer-Verlag, 2004 (cit. on p. 27).

[Log03] A. Logan. "The Kodaira dimension of moduli spaces of curves with marked points". *American Journal of Mathematics* **125** (2003), 105–138 (cit. on pp. 30–32, 66, 71, 72, 96, 97).

[Mas82] H. Masur. "Interval exchange transformations and measured foliations". *Annals of Mathematics* **115**.1 (2 1982), 169–200 (cit. on p. 18).

[MM64] T. Matsusaka and D. Mumford. "Two fundamental theorems on deformations of polarized varieties". *American Journal of Mathematics* **86** (3 1964), 668–684 (cit. on p. 83).

[MP13] D. Maulik and R. Pandharipande. "Gromov-Witten theory and Noether-Lefschetz theory". *In: A celebration of algebraic geometry, Vol. 18, Clay Math. Proc.* (2013), 469–507 (cit. on p. 34).

[MM83] S. Mori and S. Mukai. "The uniruledness of the moduli space of curves of genus 11". *Algebraic Geometry, Proc. Tokyo/Kyoto, Lecture Notes in Math.* **1016** (1983), 334–353 (cit. on pp. 33, 40, 41).

[Muk88] S. Mukai. "Curves, K3 surfaces and Fano 3-folds of genus $\leqslant 10$". *Algebraic Geometry and Commutative Algebra* **1** (1988), 357–377 (cit. on pp. 33, 41, 63, 64, 84).

[Muk92a] S. Mukai. “Fano 3-folds”. *London Math. Soc. Lecture Notes Ser.* **179** (1992), 255–263 (cit. on p. 41).

[Muk92b] S. Mukai. “Polarized K3 surfaces of genus 18 and 20”. *In: Complex Projective Geometry, London Math. Soc. Lecture Notes Ser.* **179** (1992), 264–276 (cit. on pp. 33, 64).

[Muk96] S. Mukai. “Curves and K3 surfaces of genus eleven”. *Moduli of vector bundles, Lecture Notes in Pure and Appl. Math.* **179** (1996), 189–197 (cit. on pp. 40, 41, 65, 84).

[Muk06] S. Mukai. “Polarized K3 surfaces of genus 13”. *In: Moduli Spaces and Arithmetic Geometry (Kyoto 2004), Advanced Studies in Pure Mathematics.* **45** (2006), 315–326 (cit. on pp. 33, 64).

[Muk10] S. Mukai. “Curves and Symmetric Spaces, II”. *Annals of Mathematics* **172**.3 (2010), 1539–1558 (cit. on p. 86).

[Muk12] S. Mukai. “K3 surfaces of genus 16”. *RIMS preprint 1743, 2012, available at www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1743.pdf.* (2012) (cit. on pp. 33, 64).

[Mul17] S. Mullane. “Divisorial strata of abelian differentials”. *International Mathematics Research Notices* **6** (2017), 1717–1748 (cit. on p. 86).

[Mum71] D. Mumford. “Theta characteristics of an algebraic curve”. *Annales Scientifiques de l’École Normale Supérieure* **4**.2 (1971), 181–192 (cit. on p. 9).

[Nam80] Y. Namikawa. “Toroidal compactification of Siegel spaces”. *Lecture Notes in Math.* **812** (1980) (cit. on p. 8).

[Nue16] H. Nuer. “Unirationality of moduli spaces of special cubic fourfolds and K3 surfaces”. *In: Rationality Problems in Algebraic Geometry. Lecture Notes in Mathematics. Pardini, R. and Pirola, G. (eds)* **2172** (2016) (cit. on p. 65).

[Ott88] G. Ottaviani. “Spinor bundles on quadrics”. *Trans. Amer. Math. Soc.* **307**.1 (1988), 301–316 (cit. on p. 104).

[PSY16] N. Pavic, J. Shen, and Q. Yin. “On O’Grady’s generalized Franchetta conjecture”. *Int. Math. Res. Not. (to appear)* (2016) (cit. on p. 103).

[Pet15] A. Peterson. “Modular forms on the moduli space of polarised K3 surfaces”. *PhD Thesis, University of Amsterdam* (2015) (cit. on p. 34).

[Pin74] H. C. Pinkham. “Deformations of algebraic varieties with G_m action”. *Société Mathématique de France, Paris, Astérisque No. 20* (1974) (cit. on p. 68).

[Pol06] A. Polishchuk. "Moduli spaces of curves with effective r-spin structures". *In: Gromov Witten theory of spin curves and orbifolds, Vol. 403 of Contemporary Mathematics* (2006), 1–20 (cit. on pp. 13, 21–23).

[Rei87] M. Reid. "Young person's guide to canonical singularities". *Proc. Sympos. Pure Math* **46**.1 (1987) (cit. on p. 85).

[Sau17] A. Sauvaget. "Cohomology classes of strata of differentials, arXiv: 1701.07867" (2017) (cit. on pp. 99, 102).

[Sca87] F. Scattone. "On the compactification of moduli spaces for algebraic K3 surfaces". *Mem. Amer. Math. Soc.* **70**.374 (1987) (cit. on p. 7).

[Sch16] J. Schmitt. "Dimension theory of the moduli space of twisted k-differentials, arXiv: 1607.08429" (2016) (cit. on pp. 21–23, 78).

[Seg30] B. Segre. "Sui moduli delle curve algebriche". *Annali di Matematica* **4** (1930), 71–102 (cit. on p. 30).

[Ser81] E. Sernesi. "L'unirazionalitá della varietá dei moduli delle curve di genere 12". *Ann. Scuola Normale Sup. Pisa* **8** (1981), 405–439 (cit. on p. 30).

[Ser06] E. Sernesi. *Deformations of algebraic schemes*. Grundlehren der Mathematischen Wissenschaften 334. Springer-Verlag, New York, 2006 (cit. on pp. 45, 52, 53, 69).

[Sev15] F. Severi. "Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann". *Rendiconti della R. Accad. Naz. Lincei* **24** (1915), 877–888 (cit. on p. 30).

[Sha80] J. Shah. "A complete moduli space for K3 surfaces of degree 2". *Annals of Mathematics* **112**.3 (2 1980), 485–510 (cit. on p. 8).

[Stacks] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`. 2018 (cit. on p. 96).

[Tan82] A. Tannenbaum. "Families of curves with nodes on K3 surfaces". *Math. Annalen* **260** (1982), 239–253 (cit. on p. 68).

[Tei88] M. Teixidor i Bigas. "The divisor of curves with a vanishing theta-null". *Compositio Mathematica* **66**.1 (1988), 15–22 (cit. on p. 19).

[Vee82] W. A. Veech. "Gauss measures for transformations on the space of interval exchange maps". *Annals of Mathematics* **115**.1 (2 1982), 201–242 (cit. on p. 18).

[Vee86] W. A. Veech. "Théorème de Torelli pour les cubiques de $\mathbb{P}^5$". *Inventiones Mathematicae* **86** (1986), 577–601 (cit. on p. 64).

[Vee90] W. A. Veech. "Moduli spaces of quadratic differentials". *Journal d'Analyse Mathematique* **55** (1 1990), 117–171 (cit. on p. 78).

[Ver05] A. Verra. "The unirationality of the moduli space of curves of genus $\leqslant 14$". *Compositio Mathematica* **141**.6 (2005), 1425–1444 (cit. on p. 30).

[Vie77] E. Viehweg. "Canonical divisors and the additivity of the Kodaira dimensions for morphisms of relative dimension one". *Compositio Mathematica* **3** (1977), 197–223 (cit. on p. 29).

[Vie83] E. Viehweg. "Weak positivity and the additivity of the Kodaira dimension for certain fiber spaces". *Adv. Stud. in Pure Math. Vol. 1, Alegebraic varieties and analytic varieties* (1983), 329–353 (cit. on p. 29).

[Voi02a] C. Voisin. "Green's generic syzygy conjecture for curves of even genus lying on a K3 surface". *Journal of the European Mathematical Society* **4**.4 (2002), 363–404 (cit. on p. 39).

[Voi02b] C. Voisin. *Hodge theory and complex algebraic geometry*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002 (cit. on p. 4).

[Voi05] C. Voisin. "Green's canonical syzygy conjecture for generic curves of odd genus". *Comp. Math.* **141**.5 (2005), 1163–1190 (cit. on p. 39).

[Wlo05] J. Wlodarczyk. "Simple Hironaka resolution in characteristic zero". *Journal of the American Math. Soc.* **18**.4 (2005), 779–822 (cit. on p. 27).

[Wri15] A. Wright. "Translation surfaces and their orbit closures: an introduction for a broad audience". *EMS Surv. Math. Sci.* **2**.1 (2015), 63–108 (cit. on pp. 16, 18).

Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 26. März 2018 Ignacio Barros